Lévy diffusion in a force field, Huber relaxation kinetics, and nonequilibrium thermodynamics: *H* theorem for enhanced diffusion with Lévy white noise

Marcel O. Vlad,[1,3] John Ross,[1] and Friedemann W. Schneider[2]
[1]Department of Chemistry, Stanford University, Stanford, California 94305-5080
[2]University of Würzburg, Institute of Physical Chemistry, Am Hubland, D-97074 Würzburg, Germany
[3]Center of Mathematical Statistics, Casa Academiei Romane, Calea Septembrie 13, Bucharest 5, Romania
(Received 23 February 2000; revised manuscript received 24 April 2000)

A characteristic functional approach is suggested for Lévy diffusion in disordered systems with external force fields. We study the overdamped motion of an ensemble of independent particles and assume that the force acting upon one particle is made up of two additive components: a linear term generated by a harmonic potential and a second term generated by the interaction with the disordered system. The stochastic properties of the second term are evaluated by using Huber's approach to complex relaxation [Phys. Rev. B 31, 6070 (1985)]. We assume that the interaction between a moving particle and the environment can be expressed by the contribution of a large number of relaxation channels, each channel having a very small probability of being open and obeying Poisson statistics. Two types of processes are investigated: (a) Lévy diffusion with static disorder for which the fluctuations of the random force are frozen and last forever and (b) diffusion with strong dynamic disorder and independent Lévy fluctuations (Lévy white noise). In both cases we show that the probability distribution of the position of a diffusing particle tends towards a stationary nonequilibrium form. The characteristic functional of concentration fluctuations is evaluated in both cases by using the theory of random point processes. For large times the fluctuations of the concentration field are stationary and the corresponding probability density functional can be evaluated analytically. In this limit the fluctuations depend on the distribution of the total number of particles but are independent of the initial positions of the particles. We show that the logarithm of the stationary probability functional plays the role of a nonequilibrium thermodynamic potential, which has a structure similar to the Helmholtz free energy in equilibrium thermodynamics: it is made up of the sum of an energetic component, depending on the external mechanical potential, and of an entropic component, depending on the concentration field. We show that the conditions for the existence and stability of the nonequilibrium steady state, which emerges for large times, can be expressed in terms of the stochastic potential. For Lévy white noise the average concentration field can be expressed as the solution of a fractional Fokker-Planck equation. We show that the stochastic potential is a Lyapunov function of the fractional Fokker-Planck equation, which ensures that all transient solutions for the average concentration field tend towards a unique stationary form.

PACS number(s): 05.40.−a, 64.60.Ht, 05.70.Ln, 68.35.Fx

I. INTRODUCTION

The study of Lévy diffusion in condensed matter started over 20 years ago with the classical papers by Eliot Montroll and his co-workers on disordered systems [1]. In the meantime the study of this subject has become an active area of applied statistical physics. The study of Lévy diffusion is of interest in connection with a large class of phenomena from physics, chemistry, and biology, ranging over the study of vortex motion in high-temperature superconductors, moving interfaces in porous media, random field magnets, spin glasses, the propagation of electromagnetic or acoustic waves in random media [2–4], random-phase modulation in spectroscopy [5], reaction kinetics in disordered systems [6,7], the structure of biological organs [8], or the growth of a population in a random environment [9].

There are two different types of Lévy diffusion; both types were introduced in the original papers by Montroll and collaborators [1]. The first type corresponds to a continuous time random walk for which the probability density of the waiting time between two jumps is of the Lévy type and has infinite moments. In this case the probability density of the position of the moving particle is not of the Lévy type and the moments of the position are generally finite. This type of process has been extensively investigated in the literature. A second type of process corresponds to a random walk for which the length of an individual jump obeys Lévy statistics and the moments of the displacement vector are infinite. In this second case, it has been pointed out in the literature [10] that for a particle with a finite mass the Lévy picture is wrong for most physical transport models, for which a particle must have a finite velocity of propagation. With a few exceptions [11], until recently little attention has been paid to this second type of process. A systematic study of random walks with individual jumps obeying Lévy statistics has been initiated by Fogedby [12]. He pointed out correctly that for these processes the classical approach for studying diffusion in terms of the Langevin-Einstein approach breaks down because both the second moment of the position of a moving particle as well as its average kinetic energy are divergent. In order to overcome these difficulties he has developed an alternative description based on the use of fractional calculus and of the renormalization group [12,13]. An interesting approach is based on the use of a fractional Fokker-Planck equation [14].

It has been recently pointed out that [13] in the presence of a force field, Lévy diffusion leads, in the limit of large time, to nonequilibrium stationary profiles, which have a stable Lévy shape and do not obey Maxwell-Boltzmann statistics. The emergence of nonequilibrium distributions for large times seems to be a generic feature of disordered systems. A similar nonequilibrium distribution has been found by us in the case of reversible rate processes in systems with dynamic disorder [15].

The purpose of the present paper is to investigate the possibilities of building a thermodynamic formalism for these types of nonequilibrium states based on the study of concentration fluctuations. We compute the probability density functional for a system with Lévy diffusion and define a stochastic potential proportional to the logarithm of this probability density functional. We are going to show that this nonequilibrium potential may serve as the basis for developing a nonequilibrium thermodynamic formalism which is a generalization of the thermodynamic and stochastic theory of nonequilibrium processes by Ross, Hunt, and Hunt [16–20].

Our approach is based on two different mathematical techniques. The first technique is the random-channel approach for the study of independent fluctuations in systems with static disorder introduced by Huber [21] in 1985 in the context of the theory of relaxation. Huber's approach has been recently generalized for interacting fluctuations as well as for systems with dynamical disorder [22–24]. The second technique is the characteristic functional approach to the theory of random point processes [25–27].

The structure of the paper is the following. In Sec. II we give a general formulation of the problem of Lévy diffusion based on the use of the Huber approach. Sections III and IV deal with the one-particle formulation for systems with static disorder and for systems with Lévy white noise, respectively. In Secs. V and VI many-body formulations of the theory are presented both for static disorder and for systems with Lévy white noise. In Sec. VII the many-body approach is used for extending the thermodynamic and stochastic theory of nonequilibrium processes by Ross, Hunt, and Hunt to the case of Lévy diffusion.

II. FORMULATION OF THE PROBLEM

We study the motion of a large number of identical particles in a disordered system and assume that these particles do not interact with each other. They only interact with the environment and with an external potential field. In general the equation of motion of a particle can be expressed by a generalized Langevin equation of the type

$$m\frac{d^2}{dt^2}\mathbf{r}(t)+\int_{t_0}^{t}\gamma(t-t')\cdot\frac{d}{dt'}\mathbf{r}(t')dt' = \mathbf{F}_{\mathrm{th}}(t)+\mathbf{F}_{\mathrm{dis}}(t)-\nabla U(\mathbf{r}), \tag{1}$$

where $\mathbf{r}(t)$ is the position vector of the particle at time t, m is the mass of the particle, $\gamma(t-t')$ is the tensor of friction coefficients, $\mathbf{F}_{\mathrm{th}}(t)$ is a thermal random force responsible for ordinary diffusion, $\mathbf{F}_{\mathrm{dis}}(t)$ is a random force expressing the interaction of the particle with the random medium and $U(\mathbf{r})$ is a potential field. For simplicity in this article we limit ourselves to the study of one-dimensional diffusion and assume that the motion of particles is overdamped, that is, the inertial term in Eq. (1) can be neglected. We also assume that the friction is without delay and that the random force $\mathbf{F}_{\mathrm{dis}}(t)$ generated by the random environment is much bigger than the thermal random force $\mathbf{F}_{\mathrm{th}}(t)$ responsible for the ordinary diffusion. We also assume that the mechanical potential $U(\mathbf{r})$ is harmonic. Under these circumstances the evolution equation of a moving particle reduces to

$$\gamma\frac{d}{dt}x(t)=F_{\mathrm{dis}}(t)-\frac{\partial}{\partial x}U(x), \tag{2}$$

with

$$U(x)=\tfrac{1}{2}kx^2. \tag{3}$$

The fluctuations of the random force $F_{\mathrm{dis}}(t)$ are described by using a generalized Huber approach. The random force $F_{\mathrm{dis}}(t)$ is made up of the additive contribution of a large number N of individual components. Each individual component $g_m(t)$ is a random function corresponding to a given relaxation channel

$$F_{\mathrm{dis}}(t)=\sum_{m=1}^{N} g_m(t). \tag{4}$$

The physical model for the Lévy diffusion is essentially a model of Brownian motion for which the contribution of an individual event (a collision) to the random force is a stochastic function, which obeys fractal statistics. The statistics of individual events can be conveniently expressed in terms of the Huber's theory of complex relaxation [21–24]; in this context we formally attach a channel to each individual (collision) event. Both the number N of channels and the contributions $g_1(t),\ldots,g_N(t)$ of the different channels are random; their stochastic properties can be described by a set of grand canonical probability density functionals

$$Q_0, Q_1[g_1(t)]\mathcal{D}[g_1(t)],\ldots,Q_N[g_1(t),\ldots,g_N(t)] \times\mathcal{D}[g_1(t)]\cdots\mathcal{D}[g_N(t)], \tag{5}$$

where $\mathcal{D}[g(t)]$ is a suitable integration measure over the space of functions $g(t)$. These probability density functionals obey the normalization condition

$$Q_0+\sum_{N=1}^{\infty}\frac{1}{N!}\iint\cdots\iint Q_N[g_1(t),\ldots,g_N(t)] \times\mathcal{D}[g_1(t)]\cdots\mathcal{D}[g_N(t)]=1, \tag{6}$$

where $\iint$ denotes the operation of path integration. In this paper we assume that all relaxation channels are independent and thus the grand canonical probability density functionals are Poissonian

$$Q_0=\exp\left\{-\iint\rho_g[g_1(t)]\mathcal{D}[g_1(t)]\right\}, \tag{7}$$

$$Q_N[g_1(t),\ldots,g_N(t)]\mathcal{D}[g_1(t)]\cdots\mathcal{D}[g_N(t)] = \exp\left\{-\iint \rho_g[g_1(t)]\mathcal{D}[g_1(t)]\right\} \times \rho_g[g_1(t)]\mathcal{D}[g_1(t)]\cdots\rho_g[g_N(t)]\mathcal{D}[g_N(t)], \tag{8}$$

where $\rho_g[g(t)]\mathcal{D}[g(t)]$ is the average functional density of states corresponding to a relaxation channel.

In this paper we are interested in two different problems. The first problem is the evaluation of the probability $P(x;t)dx$ that a moving particle is at a position between x and $x+dx$ at time t. The second problem is the evaluation of the stochastic properties of the concentration of particles at position x and time t. Solving these problems involves the evaluation of ensemble averages in terms of the grand canonical probability densities (7) and (8).

The probability $P(x;t)dx$ can be expressed as the average of a δ function

$$P(x;t)=\langle \delta(x-x[x(t_0);F_{\rm dis}])\rangle, \tag{9}$$

where $x[x(t_0);F_{\rm dis}]$ is the solution of the Langevin equation (2) for a given realization of the random force $F_{\rm dis}(t)$ and the average $\langle\cdots\rangle$ is computed by using the grand canonical probability density functionals (7) and (8).

The probability density functional of the concentration field $C(x;t)$,

$$\mathcal{P}[C(x;t)]\mathcal{D}[C(x;t)] \quad \text{with} \quad \iint \mathcal{P}[C(x;t)]\mathcal{D}[C(x;t)]=1 \tag{10}$$

can be expressed as the ensemble average of a δ functional

$$\delta[C(x;t)-C[x;t;F^{(1)}_{\rm dis}(t'),F^{(2)}_{\rm dis}(t'),\ldots]] \times\mathcal{D}[C(x;t)], \tag{11}$$

where $C[x;t;F^{(1)}_{\rm dis}(t'),F^{(2)}_{\rm dis}(t'),\ldots]$ is one realization of the concentration field, which can be expressed as the sum of δ functions, each δ function corresponding to one particle,

$$C[x;t;F^{(1)}_{\rm dis}(t'),F^{(2)}_{\rm dis}(t'),\ldots]=\sum_u \delta(x-x[x_u(t_0);F^{(u)}_{\rm dis}]). \tag{12}$$

We have

$$\mathcal{P}[C(x;t)]\mathcal{D}[C(x;t)] = \left\langle \delta\left(C(x;t)-\sum_u \delta(x-x[x_u(t_0);F^{(u)}_{\rm dis}])\right) \times\mathcal{D}[C(x;t)]\right\rangle, \tag{13}$$

where, once again, the average $\langle\cdots\rangle$ is computed by using the grand canonical probability density functionals (7) and (8).

The evaluation of the ensemble averages (9) and (13) for arbitrary fluctuations of the environment is in general very difficult. In this paper we limit ourselves to two different extreme cases. The first case corresponds to systems with static disorder for which the fluctuations of the random force are frozen, that is, a fluctuation once it occurs, lasts forever. The second case corresponds to the other extreme, of rapid independent fluctuations of the random force; that is, to fluctuations with Lévy white noise.

III. SYSTEMS WITH STATIC DISORDER. ONE-PARTICLE DESCRIPTION

For systems with static disorder the contributions $g_1, g_2, \ldots$ of different relaxation channels to the random force are random numbers, rather than random functions, and the grand canonical probability density functionals (5) are replaced by probability densities

$$Q_0, Q_1(g_1)dg_1,\ldots,Q_N(g_1,\ldots,g_N)dg_1\cdots dg_N, \tag{14}$$

with the normalization condition

$$Q_0+\sum_{N=1}^{\infty}\frac{1}{N!}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}Q_N(g_1,\ldots,g_N)dg_1\cdots dg_N=1. \tag{15}$$

The grand canonical probability densities (14) describe the statistics of the individual events (channels). Later on, we shall introduce a different set of grand canonical probability densities, which describe the concentration fluctuations. These two sets of probability densities are distinct and should not be mistaken. For independent relaxation channels the probability densities $Q_1, Q_2, \ldots$ are Poissonian

$$Q_0=\exp\left(-\int_{-\infty}^{+\infty}\rho_g(g)dg\right), \tag{16}$$

$$Q_N(g_1,\ldots,g_N)dg_1\cdots dg_N = \exp\left(-\int_{-\infty}^{+\infty}\rho_g(g)dg\right)\rho_g(g_1)dg_1\cdots\rho_g(g_N)dg_N, \tag{17}$$

where $\rho_g(g)dg$ is the average number of channels with a contribution to the random force between g and $g+dg$. For Lévy fluctuations the distribution of the numbers of channels with respect to their contribution to the random force is given by a self-similar law of the negative power-law type. For the case of relaxation theory the contribution of a channel is a positive number. A random force, however, can be either positive or negative and its distribution is usually symmetrical. For this reason we assume that the average distribution of the number of channels is given by a negative power law in the absolute value of the contribution g to the random force

$$\rho_g(g)dg=\kappa|g|^{-(1+\alpha)}dg, \quad \kappa>0, \quad 2>\alpha>0 \tag{18}$$

where $\kappa>0$ is a positive proportionality coefficient and $2>\alpha>0$ is a positive fractal exponent. For the particular case of the harmonic potential (3) the Langevin equation (2) becomes

$$\gamma\frac{d}{dt}x(t)+kx(t)=F_{\mathrm{dis}}(t). \tag{19}$$

Equation (19) has the solution

$$x(t)=x(t_0)\exp\left[-\frac{k}{\gamma}(t-t_0)\right]+\frac{1}{\gamma}\int_{t_0}^{t}\exp\left[-\frac{k}{\gamma}(t-t')\right]F_{\mathrm{dis}}(t')dt'. \tag{20}$$

For static (quenched) disorder F_{dis} is independent of time and

$$x(t)=x(t_0)\exp\left[-\frac{k}{\gamma}(t-t_0)\right]+\frac{1}{k}\left[1-\exp\left(-\frac{k}{\gamma}(t-t_0)\right)\right]F_{\mathrm{dis}}. \tag{21}$$

We use the notation

$$G(q;t)=\int_{-\infty}^{+\infty}P(x;t)\exp[ixq]dx \tag{22}$$

for the Fourier transform (the characteristic function) of the probability density $P(x;t)$ of the position x of a particle at time t. We assume that the fluctuations of the initial position of a particle $x(t_0)$ are independent of the fluctuations of the random force F_{dis} and use Eqs. (9) and (22) for expressing $G(q;t)$ as a grand canonical average

$$\begin{aligned}G(q;t)&=\langle\exp[ix(t)q]\rangle\\&=\left\langle\exp\left(iq\left\{x(t_0)\exp\left(-\frac{k}{\gamma}(t-t_0)\right)+\frac{1}{k}\left[1-\exp\left(-\frac{k}{\gamma}(t-t_0)\right)\right]F_{\mathrm{dis}}\right\}\right)\right\rangle\\&=G_{\mathrm{transient}}(q;t)G_{\mathrm{normal}}(q;t),\end{aligned} \tag{23}$$

where

$$G_{\mathrm{transient}}(q;t)=\left\langle\exp\left\{iq\left[x(t_0)\exp\left(-\frac{k}{\gamma}(t-t_0)\right)\right]\right\}\right\rangle=G_0\left(q\exp\left[-\frac{k}{\gamma}(t-t_0)\right]\right), \tag{24}$$

$$G_{\mathrm{normal}}(q;t)=Q_0+\sum_{N=1}^{\infty}\frac{1}{N!}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}Q_N(g_1,\ldots,g_N)dg_1\cdots dg_N\exp\left\{iq\frac{1}{k}\left[1-\exp\left(-\frac{k}{\gamma}(t-t_0)\right)\right]\sum_{m=1}^{N}g_m\right\}, \tag{25}$$

and

$$G_0(q)=\int_{-\infty}^{+\infty}\exp(iqx)P(x;t_0)dx \tag{26}$$

is the Fourier transform (characteristic function) of the probability density $P(x;t_0)$ of the initial position of the particle at time t_0. For a Poissonian process described by Eqs. (16) and (17) the sum in Eq. (25) over the number N of channels can be easily evaluated, resulting in

$$G_{\mathrm{normal}}(q;t)=\exp\left[-\int_{-\infty}^{+\infty}\rho_g(g)\left(1-\exp\left\{iq\frac{1}{k}\left[1-\exp\left(-\frac{k}{\gamma}(t-t_0)\right)\right]g\right\}\right)dg\right]. \tag{27}$$

If the distribution of the average number of channels is given by the self-similar law (18) then the integral in the exponent of Eq. (27) can be computed analytically. We have

$$G_{\mathrm{normal}}(q;t)=\exp\left(-\frac{2}{\alpha}\kappa\left\{|q|\frac{1}{k}\left[1-\exp\left(-\frac{k}{\gamma}(t-t_0)\right)\right]\right\}^{\alpha}\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)\right). \tag{28}$$

The inverse Fourier transform of $G_{\mathrm{normal}}(q;t)$

$$\mathcal{G}(x;t-t_0)=\mathcal{F}^{-1}G_{\mathrm{normal}}(q;t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp[-iqx]\exp\left(-\frac{2}{\alpha}\kappa^{\alpha}\left\{|q|\frac{1}{k}\left[1-\exp\left(-\frac{k}{\gamma}(t-t_0)\right)\right]\right\}^{\alpha}\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)\right)dq \tag{29}$$

is a Green function (propagator) for the diffusion process, which can be expressed in terms of the symmetric stable Lévy probability density with a fractal exponent α [28]

$$\Psi_\varepsilon(y)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp[-iqy-|y|^\alpha]dy \quad \text{with} \quad 2>\alpha>0. \tag{30}$$

We have

$$\mathcal{G}(x;t-t_0)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp[-iqx] \times\exp\{-[|q|\zeta(t-t_0)]^\alpha\}dq =\frac{1}{\zeta(t-t_0)}\Psi_\alpha[x/\zeta(t-t_0)], \tag{31}$$

where

$$\zeta(t-t_0)=\left\{\frac{2}{\alpha}\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)\right\}^{1/\alpha} \times\kappa^{1/\alpha}\frac{1}{k}\left[1-\exp\left(-\frac{k}{\gamma}(t-t_0)\right)\right]. \tag{32}$$

For large times this propagator tends towards a nonequilibrium stationary form. We have

$$\lim_{t\to\infty}\mathcal{G}(x;t-t_0)=\frac{1}{\zeta_\infty}\Psi_\alpha\left(\frac{x}{\zeta_\infty}\right), \tag{33}$$

with

$$\zeta_\infty=\left\{\frac{2}{\alpha}\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)\right\}^{1/\alpha}\frac{1}{k}\kappa^{1/\alpha}. \tag{34}$$

The probability density $P(x;t)$ of the position of a particle at time t can be computed from Eqs. (22)–(24) and (28). We obtain

$$p(x;t)=\mathcal{F}^{-1}[G(q;t)] =\int_{-\infty}^{+\infty}\frac{P_0(y')}{\zeta_\infty}\Psi_\alpha\left\{\frac{x-y'\xi(t-t_0)}{\zeta_\infty\{1-\xi(t-t_0)\}}\right\}dy' =\int_{-\infty}^{+\infty}\frac{P_0(y')}{\zeta_\infty}\Psi_\alpha\left\{\frac{x\exp[(t-t_0)k/\gamma]-y'}{\zeta_\infty\{\exp[(t-t_0)k/\gamma]-1\}}\right\}dy' \tag{35}$$

where

$$\xi(t-t_0)=\exp[-(t-t_0)k/\gamma]. \tag{36}$$

From Eq. (35) it is easy to check that for large times the probability density $P(x;t)$ tends towards a nonequilibrium stationary form which is independent of the initial probability density $P(x;t_0)$ of the position of a particle at time t_0. We have

$$\lim_{t\to\infty}P(x;t)=\lim_{t\to\infty}\mathcal{G}(x;t-t_0) =\frac{1}{\zeta_\infty}\Psi_\alpha\left(\frac{x}{\zeta_\infty}\right) \quad \text{independent of} \quad P(x;t_0). \tag{37}$$

In conclusion, in this section we have shown that for systems with static disorder and a harmonic potential, the probability density $P(x;t)$ of the position of a moving particle can be represented by a convolution product between the initial probability density $P(x;t_0)$ at time t_0 and a propagator $\mathcal{G}(x;t-t_0)$, which is given by a time-dependent stable probability density of the Lévy type. For large times both the state probability density $P(x;t)$ and the propagator $\mathcal{G}(x;t-t_0)$ tend towards a time-independent nonequilibrium probability density of the Lévy type which is independent of the initial state of the system.

IV. SYSTEMS WITH LÉVY WHITE NOISE. ONE-PARTICLE DESCRIPTION

For systems with Lévy white noise the grand canonical probability density functionals can be expressed as the continuous limit of a product of static probability densities of the type (14). We divide the time interval of length $t-t_0$ into small slices of length Δt. For Lévy white noise the fluctuations of a g variable attached to one channel at a given time is independent of the fluctuations at other times; we have

$$Q_N[g_1(t),\ldots,g_N(t)]\mathcal{D}[g_1(t)]\cdots\mathcal{D}[g_N(t)] =\lim_{\Delta t\to0}\left\{\prod_u\left[\exp\left(-\int_{-\infty}^{+\infty}\rho_g(g^{(u)};\Delta t)dg^{(u)}\right) \times\rho_g(g_1^{(u)})dg_1^{(u)}\cdots\rho_g(g_N^{(u)})dg_N^{(u)}\right]\right\}, \tag{38}$$

where

$$g_n^{(u)}=g_n^n(t_0+u\Delta t), \quad n=1,\ldots,N. \tag{39}$$

The average density of channels $\rho_g(g;\Delta t)$ is given by a scaling law similar to Eq. (18), where now the proportionality coefficient κ is a function of the length Δt of the time interval

$$\rho_g(g;\Delta t)dg=\kappa(\Delta t)|g|^{-(1+\alpha)}dg, \quad \kappa>0, \quad 2>\alpha>0. \tag{40}$$

In order that the averages computed in terms of the grand canonical probability density functionals (38) are physically consistent it is necessary that $\kappa(\Delta t)$ obeys the scaling condition

$$\kappa=\kappa_0(\Delta t)^{1-\alpha} \quad \text{with} \quad \kappa_0>0. \tag{41}$$

The evaluation of the Fourier transform of the state probability density is straightforward. For averaging we discretize the expression of $x(t;x(t_0))$ compute the average and then pass to the limit $\Delta t\to0$. We have

$$
\begin{aligned}
G(q;t) &= \langle \exp[ix(t)q] \rangle \\
&= \left\langle \exp\left\{ iq\left[x(t_0)\exp\left(-\frac{k}{\gamma}(t-t_0) \right) + \frac{1}{\gamma}\int_{t_0}^{t} \exp\left[-\frac{k}{\gamma}(t-t') \right] F_{\mathrm{dis}}(t')dt' \right] \right\} \right\rangle \\
&= \left\langle \exp\left\{ iq\left[x(t_0)\exp\left(-\frac{k}{\gamma}(t-t_0) \right) \right] \right\} \right\rangle \\
&\quad \times \prod_u \left\langle \exp\left\{ iq\left[\frac{1}{\gamma}\exp\left(-\frac{k}{\gamma}(t-t'_u) \right) F_{\mathrm{dis}}(t'_u)\Delta t' \right] \right\} \right\rangle \\
&= G_{\mathrm{transient}}(q;t) \prod_u G_{\mathrm{infi}}(q;t;t'_u),
\end{aligned}
\tag{42}
$$

where

$$
G_{\mathrm{infi}}(q;t;t'_u) = \left\langle \exp\left\{ iq\left[\frac{1}{\gamma}\exp\left(-\frac{k}{\gamma}(t-t'_u) \right) F_{\mathrm{dis}}(t'_u)\Delta t' \right] \right\} \right\rangle,
\tag{43}
$$

and $G_{\mathrm{transient}}(q;t)$ is given by Eq. (24). By comparing Eqs. (24) and (43) we notice that $G_{\mathrm{infi}}(q;t;t'_u)$ has a structure similar to $G_{\mathrm{normal}}(q;t)$ for systems with static disorder given by Eq. (25). If in Eq. (43) we make the replacements

$$
G_{\mathrm{infi}}(q;t;t'_u) \to G_{\mathrm{normal}}(q;t),
\tag{44}
$$

$$
\frac{1}{\gamma}\exp\left[-\frac{k}{\gamma}(t-t'_u) \right]\Delta t' \to \frac{1}{k}\left[1-\exp\left(-\frac{k}{\gamma}(t-t'_u) \right) \right],
\tag{45}
$$

we get Eq. (25). By making use of this observation we can easily evaluate $G_{\mathrm{infi}}(q;t;t'_u)$ from Eq. (28). We get

$$
G_{\mathrm{infi}}(q;t;t'_u) = \exp\left\{ -\frac{2}{\alpha}\kappa\left[q\,\frac{1}{\gamma}\exp\left(-\frac{k}{\gamma}(t-t'_u) \right)\Delta t' \right]^{\alpha} \Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2} \right) \right\}.
\tag{46}
$$

The next step is to evaluate $G_{\mathrm{normal}}(q;t)$

$$
G_{\mathrm{normal}}(q;t) = \prod_u G_{\mathrm{infi}}(q;t;t'_u)
\tag{47}
$$

by passing to the continuous limit. From Eqs. (41) and (46) and (47) we come to

$$
\begin{aligned}
G_{\mathrm{normal}}(q;t) &= \exp\left\{ -|q|^{\alpha}\frac{2\kappa_0}{\alpha\gamma^{\alpha}}\int_{t_0}^{t}\exp\left[-\frac{\alpha k}{\gamma}(t-t') \right]dt'\,\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2} \right) \right\} \\
&= \exp\left\{ -|q|^{\alpha}\frac{2\kappa_0}{\alpha^2\gamma^{\alpha-1}k}\left[1-\exp\left(-\frac{\alpha k}{\gamma}(t-t') \right) \right]\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2} \right) \right\} \\
&= \exp\{ -|q|^{\alpha}[\zeta^*(t-t_0)]^{\alpha} \},
\end{aligned}
\tag{48}
$$

where

$$
\begin{aligned}
\zeta^*(t-t_0) &= \zeta^*_{\infty}\{1-\exp[-\nu(t-t_0)]\}^{1/\alpha} \\
&= \left\{ \frac{2\kappa_0}{\alpha^2 k\gamma^{\alpha-1}}\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2} \right)\{1-\exp[-\nu(t-t_0)]\} \right\}^{1/\alpha}, \quad \nu = \alpha k/\gamma
\end{aligned}
\tag{49}
$$

$$
\zeta^*_{\infty} = \left\{ \frac{2\kappa_0}{\alpha^2 k\gamma^{\alpha-1}}\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2} \right) \right\}^{1/\alpha}.
\tag{50}
$$

The propagator of the diffusion process, $\mathcal{G}(x;t-t_0)=\mathcal{F}^{-1}G(q;t)$ and the state probability $P(x;t)$ is given by a relationship similar to Eqs. (31) and (35), respectively,

$$\mathcal{G}(x;t-t_0)=\frac{1}{\zeta^*(t-t_0)}\Psi_\alpha[x/\zeta^*(t-t_0)], \tag{51}$$

and

$$\begin{aligned}P(x;t)&=\mathcal{F}^{-1}[G(q;t)]\\&=\int_{-\infty}^{+\infty}\frac{P_0(y')}{\zeta_\infty^*}\\&\quad\times\Psi_\alpha\left\{\frac{x\exp[v(t-t_0)/\alpha]-y'}{\zeta_\infty^*\{\exp[\nu(t-t_0)]-1\}^{1/\alpha}}\right\}dy'.\end{aligned} \tag{52}$$

The time dependence of the propagator $\mathcal{G}(x;t-t_0)$ and of the state probability $P(x;t)$ is different for systems with static disorder and for systems with white Lévy noise, respectively. However, in the long run, they both tend towards the same type of nonequilibrium stationary profile. In the case of white Lévy noise, we have

$$\begin{aligned}\lim_{t\to\infty}P(x;t)&=\lim_{t\to\infty}\mathcal{G}(x;t-t_0)\\&=\frac{1}{\zeta_\infty^*}\Psi_\alpha\left(\frac{x}{\zeta_\infty^*}\right)\ \text{independent of}\ P(x;t_0).\end{aligned} \tag{53}$$

Equation (53) is similar to Eq. (37) derived in the preceding section for systems with static disorder.

We mention that the results derived in this section are consistent with the results presented in Ref. [13]. Our equations are slightly more general than the ones derived in Ref. [13] because in our derivations we have assumed arbitrary initial conditions.

V. SYSTEMS WITH STATIC DISORDER. MANY-BODY DESCRIPTION

In order to study the dynamics of concentration fluctuations we introduce a set of grand canonical probabilities for the number M and the positions $x_1,\ldots,x_M$ of the particles at time t,

$$\mathcal{C}_0,\mathcal{C}_M(x_1,\ldots,x_M;t)dx_1\cdots dx_M \tag{54}$$

with the normalization condition

$$\mathcal{C}_0+\sum_{M=1}^{\infty}\frac{1}{M!}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\mathcal{C}_M(x_1,\ldots,x_M;t)dx_1\cdots dx_M=1, \tag{55}$$

and assume that these functions are known for the initial time t_0

$$\begin{aligned}\mathcal{C}_M(x_1,\ldots,x_M;t=t_0)&dx_1\cdots dx_M\\&=\mathcal{C}_M^0(x_1,\ldots,x_M)dx_1\cdots dx_M.\end{aligned} \tag{56}$$

As mentioned in the preceding section the grand canonical probability densities, $\mathcal{C}_0,\mathcal{C}_1,\ldots$ which describe the concentration fluctuations, are different from the probability densities $Q_0,Q_1,\ldots$ which describe the fluctuations of the random force. We introduce the characteristic functional $\mathcal{G}[\mathcal{K}(x';t)]$ attached to the probability density functional $\mathcal{P}[C(x;t)]\mathcal{D}[C(x;t)]$ of concentration fluctuations,

$$\begin{aligned}\mathcal{G}[\mathcal{K}(x';t)]&=\iint\exp\left\{i\int_{-\infty}^{+\infty}C(x';t)\mathcal{K}(x';t)dx'\right\}\\&\quad\times\mathcal{P}[C(x;t)]\mathcal{D}[C(x;t)]\\&=\left\langle\exp\left\{i\int_{-\infty}^{+\infty}C(x';t)\mathcal{K}(x';t)dx'\right\}\right\rangle,\end{aligned} \tag{57}$$

where $\mathcal{K}(x;t)$ is a test function conjugated to the concentration field $C(x;t)$. By using Eqs. (13) and (57) $\mathcal{G}[\mathcal{K}(x';t)]$ can be expressed as

$$\begin{aligned}\mathcal{G}[\mathcal{K}(x';t)]&=\left\langle\sum_{M=0}^{\infty}\frac{1}{M!}\int_{-\infty}^{+\infty}\mathcal{C}_M^0(x_1^0,\ldots,x_M^0)dx_1^0\cdots dx_M^0\exp\left\{i\int_{-\infty}^{+\infty}\sum_{m=1}^{M}\delta(x-x(x_m^0;t))\mathcal{K}(x';t)dx'\right\}\right\rangle\\&=\left\langle\sum_{M=0}^{\infty}\frac{1}{M!}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\mathcal{C}_M^0(x_1^0,\ldots,x_M^0)dx_1^0\cdots dx_M^0\prod_{M=1}^{M}\exp\{i\mathcal{K}(x(x_m^0;t);t)\}\right\rangle,\end{aligned} \tag{58}$$

where

$$x_{\sigma_\rho}(x_{\sigma_\rho}^0;t)=x_{\sigma_\rho}^0\xi(t-t_0)+\frac{1}{k}[1-\xi(t-t_0)]F_{\rm dis} \tag{59}$$

and $\xi(t-t_0)$ is given by Eq. (36).

We use the mathematical identity

$$f(x(t))=\frac{1}{2\pi}\int \exp[-ibx(t)]db\int f(\eta)\exp(ib\eta)d\eta \tag{60}$$

which can be easily derived by using a pair of direct and inverse Fourier transforms. By combining Eqs. (58) and (60) we get

$$\mathcal{G}[\mathcal{K}(x')]=\langle \mathcal{I}[\mathcal{K}(x')]\rangle, \tag{61}$$

with

$$\mathcal{I}[\mathcal{K}(x')]=\sum_{M=0}^{\infty}\frac{1}{M!}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\mathcal{C}_M^0(x_1^0,\ldots,x_M^0)dx_1^0\cdots dx_M^0$$
$$\times\frac{1}{(2\pi)^M}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\exp\left[-i\sum_u b_u x(x_u^0;t)\right]db_1\cdots db_M\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_{m=1}^{M}\exp\{i\mathcal{K}(\eta_m)\}d\eta_1\cdots d\eta_M. \tag{62}$$

Now we change the order of the integrals and express the average with respect to the random force in terms of the grand canonical probability densities. After lengthy calculations we obtain

$$\mathcal{I}[\mathcal{K}(x')]=\sum_{M=0}^{\infty}\frac{1}{M!}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\mathcal{C}_M^0(x_1^0,\ldots,x_M^0)\exp\left[-i\sum_u b_u x(t_0)\xi\right]dx_1^0\cdots dx_M^0$$
$$\times\frac{1}{(2\pi)^M}\exp\left[-i\sum_u b_u\frac{1}{k}[1-\xi]F_{\mathrm{dis}}\right]db_1\cdots db_M\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_{m=1}^{M}\exp\{i\mathcal{K}(\eta_m)\}d\eta_1\cdots d\eta_M$$
$$=\sum_{M=0}^{\infty}\frac{1}{M!}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\mathcal{M}_M(-b_1\xi,\ldots,-b_M\xi;t_0)db_1\cdots db_M$$
$$\times\frac{1}{(2\pi)^M}\exp\left[-i\sum_u b_u\frac{1}{k}[1-\xi]F_{\mathrm{dis}}\right]\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_{m=1}^{M}\exp\{i\mathcal{K}(\eta_m)+ib_m\eta_m\}d\eta_1\cdots d\eta_M, \tag{63}$$

where

$$\mathcal{M}_M^0(q_1,\ldots,q_M)=\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\exp\left(i\sum_{M=1}^{M}x_u q_u\right)\mathcal{C}_M^0(x_1,x_2,\ldots,x_M) \tag{64}$$

is the multiple Fourier transform of the grand canonical probability density $\mathcal{C}_M^0(x_1,x_2,\ldots,x_M)$. In Eq. (63), for simplicity, we have used the notation ξ for $\xi(t-t_0)$. The next step is to evaluate the average over the random force, resulting in

$$\mathcal{G}[\mathcal{K}(x')]=\sum_{M=0}^{\infty}\frac{1}{M!}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\mathcal{M}_M(-b_1\xi,\ldots,-b_M\xi;t_0)db_1\cdots db_M$$
$$\times\frac{1}{(2\pi)^M}\exp\left\{-[\zeta(t-t_0)]^\alpha\sum_{m=1}^{M}|b_m|^\alpha\right\}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_{m=1}^{M}\exp\{i\mathcal{K}(\eta_m)+ib_m\eta_m\}d\eta_1\cdots d\eta_M. \tag{65}$$

Equation (65) is the main result of this section; it contains all information concerning the fluctuations of the concentration field $C(x;t)$, expressed in terms of the different moments $\langle C(x_1;t)C(x_2;t)\cdots\rangle$ or cumulants $\langle\langle C(x_1;t)C(x_2;t)\cdots\rangle\rangle$, the grand canonical densities $\mathcal{C}_M(x_1,x_2,\ldots,x_M;t)$ as well as the corresponding multiparticle product densities $\mathcal{I}_m(x_1,x_2,\ldots,x_m;t)$ and the correlation functions $g_m(x_1,x_2,\ldots,x_m;t)$. The main steps of the mathematical derivations are outlined in Appendixes A and B. In the following we give only the results.

The moments $\langle C(x_1;t)C(x_2;t)\cdots\rangle$ and the cumulants $\langle\langle C(x_1;t)C(x_2;t)\cdots\rangle\rangle$ can be expressed as functional derivatives of $\mathcal{G}[\mathcal{K}(x')]$:

$$\langle C(x_1;t)\cdots C(x_{\mathcal{N}};t)\rangle=(i)^{-\mathcal{N}}\frac{\delta^{\mathcal{N}}}{\delta\mathcal{K}(x_1;t)\cdots\delta\mathcal{K}(x_{\mathcal{N}};t)}\mathcal{G}[\mathcal{K}(x';t)]\bigg|_{\mathcal{K}(x';t)=0}, \tag{66}$$

$$\langle\langle C(x_1;t)\cdots C(x_{\mathcal{N}};t)\rangle\rangle=(i)^{-\mathcal{N}}\frac{\delta^{\mathcal{N}}}{\delta\mathcal{K}(x_1;t)\cdots\delta\mathcal{K}(x_{\mathcal{N}};t)}\ln\mathcal{G}[\mathcal{K}(x';t)]\bigg|_{\mathcal{K}(x';t)=0}. \tag{67}$$

For moments we have managed to derive a general formula

$$\langle C(x_1;t)\cdots C(x_{\mathcal{N}};t)\rangle=\sum_{M=\mathcal{N}}^{\infty}\frac{1}{M!}\sum_{\nu_1=1}^{M}\cdots\sum_{\nu_{\mathcal{N}}=1}^{M}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\mathcal{B}_M^0(0,\ldots,-b_{\nu_1}\xi,\ldots,0,\ldots,-b_{\nu_x}\xi,\ldots,0,\ldots)$$
$$\times\frac{1}{(2\pi)^{\mathcal{N}}}\exp\left\{-\left[\zeta((t-t_0)]^{\alpha}\sum_{m=1}^{M}|b_m|^{\alpha}\right\}\exp\left(i\sum_{u=1}^{\mathcal{N}}b_{\nu_u}x_u\right)db_{\nu_1}\cdots db_{\nu_x}. \tag{68}$$

The cumulants are more difficult to evaluate. A general formula similar to Eq. (68) is not available; however, they can be computed step by step,

$$\langle\langle C(x;t)\rangle\rangle=\int_{-\infty}^{+\infty}\frac{\langle\langle C_0(x)\rangle\rangle}{\zeta_\infty}\Psi_\alpha\left\{\frac{x\exp[(t-t_0)k/\gamma]-y'}{\zeta_\infty\{\exp[(t-t_0k/\gamma]-1\}}\right\}dy', \tag{69}$$

$$\langle\langle C(x_1;t)C(x_2;t)\rangle\rangle=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\langle\langle C_0(y_1')C_0(y_2')\rangle\rangle-\delta(y_1'-y_2')\langle\langle C_0(y_1')\rangle\rangle}{(\zeta_\infty)^2}$$
$$\times\prod_{u=1,2}\Psi_\alpha\left\{\frac{x_u\exp[(t-t_0)k/\gamma]-y_u'}{\zeta_\infty\{\exp[(t-t_0)k/\gamma]-1\}}\right\}dy_1'dy_2'$$
$$+\delta(x_1-x_2)\int_{-\infty}^{+\infty}\frac{\langle\langle C_0(y_1')\rangle\rangle}{\zeta_\infty}\Psi_\alpha\left\{\frac{x_1\exp[(t-t_0)k/\gamma]-y_1'}{\zeta_\infty\{\exp[(t-t_0)k/\gamma]-1\}}\right\}dy_1', \tag{70}$$

where $C_0(x)=C(x;t_0)$.

Equation (68) for the moments of the concentration field can be used for computing the grand canonical probability densities, for the numbers and the positions of the particles at time t. In Appendix A we show that

$$\mathcal{C}_M(x_1,\ldots,x_M;t)=\int_{-\infty}^{+\infty}\cdots\int_{-My}^{+\infty}\frac{\mathcal{C}_M^0(y_1',\ldots,y_M')}{(\zeta_\infty)^M}\prod_{w=1}^{M}\Psi_\alpha\left\{\frac{x_w\exp[(t-t_0)k/\gamma]-y_w'}{\zeta_\infty\{\exp[(t-t_0)k/\gamma]-1\}}\right\}dy_1'\cdots dy_N'. \tag{71}$$

Following Carruthers [29] we introduce the product densities, corresponding to different numbers of distinct particles placed at different positions as the averages of products of δ functions,

$$\mathcal{I}_{m,}(x_1,\ldots,x_m;t)=\langle\delta(x_1-x_{\beta_1})\cdots\delta(x_1-x_{\beta_m})\rangle, \tag{72}$$

where all labels $\beta_1,\ldots,\beta_m$ are distinct. These functions can be computed by means of a chain of relationships similar to Eq. (71),

$$\mathcal{I}_m(x_1,\ldots,x_m;t)=\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\frac{\mathcal{I}_m^0(y_1',\ldots,y_m')}{(\zeta_\infty)^m}\times\prod_{w=1}^{m}\Psi_\alpha\left\{\frac{x_w\exp[(t-t_0)k/\gamma]-y_w'}{\zeta_\infty\{\exp[(t-t_0)k/\gamma]-1\}}\right\}dy_1'\cdots dy_m', \tag{73}$$

where

$$\mathcal{I}_m^0(y_1',\ldots,y_m')=\mathcal{I}_m(y_1',\ldots,y_m';t_0). \tag{74}$$

The correlation functions $\mathscr{g}_m(x_1,x_2,\ldots,x_m;t)$ are defined as the cumulants corresponding to the product densities $\mathcal{I}_m(x_1,\ldots,x_m;t)$ and can be computed starting from Eq. (73) (see Appendix B). We have

$$\mathscr{g}_m(x_1,\ldots,x_m;t)=\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\frac{\mathscr{g}_m(y_1',\ldots,y_m';t_0)}{(\zeta_\infty)^m}\times\prod_{u=1}^{m}\Psi_\alpha\left\{\frac{x_u\exp[(t-t_0)k/\gamma]-y_u'}{\zeta_\infty\{\exp[(t-t_0)k/\gamma]-1\}}\right\}dy_1'\cdots dy_m'. \tag{75}$$

In particular, the correlation function of first order is equal to the average value of the concentration field

$$\mathscr{g}_1(x;t)=\langle\langle C(x;t)\rangle\rangle=\int_{-\infty}^{+\infty}\frac{\langle\langle C_0(x)\rangle\rangle}{\zeta_\infty}\times\Psi_\alpha\left\{\frac{x\exp[(t-t_0)k/\gamma]-y'}{\zeta_\infty\{\exp[(t-t_0)k/\gamma]-1\}}\right\}dy'. \tag{76}$$

By comparing the results of the many-body approach presented in this section with the one-particle description presented in Sec. III, we notice that these two different descriptions are consistent with each other. In the first place Eq. (69) for the first cumulant of the concentration field, $\langle\langle C(x;t)\rangle\rangle$, is similar to Eq. (35) for the probability density $P(x;t)$ of the position of a moving particle at time t, derived in Sec. III. By comparing Eqs. (35) and (69) and taking into account that the first cumulant of the concentration field is equal to the average value $\langle C(x;t)\rangle$ we notice that the two equations are equivalent to each other if we assume that

$$P(x;t)=\langle\langle C(x;t)\rangle\rangle/\langle\langle M\rangle\rangle=\langle C(x;t)\rangle/\langle M\rangle, \quad (77)$$

$$P_0(x)=\langle\langle C_0(x)\rangle\rangle/\langle\langle M\rangle\rangle=\langle C_0(x)\rangle/\langle M\rangle, \quad (78)$$

where

$$\langle M\rangle=\int_{-\infty}^{+\infty}\langle C(x;t)\rangle dx=\int_{-\infty}^{+\infty}\langle C_0(x)\rangle dx \quad (79)$$

is the average number of particles in the system.

For computing the probability density functional of concentration fluctuations we need more information concerning the initial distribution of particles in the system. In the following we consider a particular case, a grand canonical ensemble of noninteracting particles, for which the initial probability densities are given by (see Appendix C)

$$\mathcal{C}_M^0(x_1,\ldots,x_M)=\exp(-\langle\langle M\rangle\rangle)\langle C_0(x_1)\rangle\cdots\langle C_0(x_M)\rangle, \quad (80)$$

the probability $P(M)$ of the total number of particles M is Poissonian

$$P(M)=\frac{1}{M!}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\mathcal{C}_M^0(x_1,\ldots,x_M)dx_1\cdots dx_M = \frac{\langle\langle M\rangle\rangle^M}{M!}\exp(-\langle\langle M\rangle\rangle), \quad (81)$$

the characteristic functional $\mathcal{G}[\mathcal{K}(x')]$ of the concentration field is given by

$$\mathcal{G}[\mathcal{K}(x')]=\exp\left(\int_{-\infty}^{+\infty}\{\exp[i\mathcal{K}(x')]-1\}\langle C(x';t)\rangle dx'\right), \quad (82)$$

and the probability density functional of concentration fluctuations, $\mathcal{P}[C(x;t)]\mathcal{D}[C(x;t)]$, can be expressed as

$$\mathcal{P}[C(x;t)]\mathcal{D}[C(x;t)] = \lim_{\forall\Delta x_u\to 0}\prod_u\left\{\frac{[\langle C(x_u;t)\rangle\Delta x_u]^{C(x_u;t)\Delta x_u}}{[C(x_u;t)\Delta x_u]!}\times\exp[-\langle C(x_u;t)\rangle\Delta x_u]\right\} = \exp\{-\Phi[C(x;t)]\}\mathcal{D}[C(x;t)], \quad (83)$$

where

$$\Phi[C(x;t)]=\int_{-\infty}^{+\infty}C(x;t)\ln\left[\frac{C(x;t)}{\langle C(x;t)\rangle}\right]dx \quad (84)$$

is a stochastic potential and the integration measure $\mathcal{D}[C(x;t)]$ is given by

$$\mathcal{D}[C(x;t)]=\lim_{\forall\Delta x_u\to 0}\prod_u\left\{\frac{\Delta[C(x_u;t)\Delta x_u]}{\sqrt{2\pi\langle C(x_u;t)\rangle\Delta x_u}}\right\}. \quad (85)$$

The field described by Eqs. (82)–(85) is Poissonian, and, as expected for a Poissonian field, all cumulants of order bigger than one are δ correlated,

$$\langle\langle C(x_1;t)C(x_2;t)\cdots C(x_m;t)\rangle\rangle = \langle\langle C(x_1;t)\rangle\rangle\prod_{u=2}^{m}\delta(x_u-x_1). \quad (86)$$

The product densities are factorizable and all correlation functions of order bigger than one are equal to zero

$$\mathcal{I}_m(x_1,\ldots,x_m;t)=\prod_{u=1}^{m}\mathcal{I}_1(x_u;t)=\prod_{u=1}^{m}\langle\langle C(x_u;t)\rangle\rangle, \quad (87)$$

$$\mathscr{g}_m(x_1,\ldots,x_m;t)=0,\quad m\geqslant 2. \quad (88)$$

In conclusion, in this section we have developed a general method for computing the characteristic functional of the concentration field, the grand canonical joint probability densities, product densities, and correlation functions for Lévy diffusion in systems with static disorder. The results derived in this section are going to be used in Sec. VII for developing a thermodynamic theory for Lévy diffusion.

VI. SYSTEMS WITH LÉVY WHITE NOISE. MANY-BODY DESCRIPTION

For systems with Lévy noise the computations follow essentially the same steps as for systems with static disorder. The expression (58) for the characteristic functional $\mathcal{G}[\mathcal{K}(x';t)]$ of the concentration field remains valid with the difference that the functions $x_{\sigma_\rho}(x_{\sigma_\rho}^0;t)$ are given by

$$x(x_{\sigma_\rho}^0;t)=x_{\sigma_\rho}^0\xi(t-t_0)+\frac{1}{\gamma}\int_{t_0}^{t}\xi(t-t')F(t')dt'. \quad (89)$$

By following the same steps as in Sec. V we can derive an expression for the ensemble average in Eq. (58),

$$\mathcal{G}[\mathcal{K}(x')]=\sum_{M=0}^{\infty}\frac{1}{M!}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\mathcal{M}_M(-b_1\xi,\ldots,-b_M\xi;t_0)db_1\cdots db_M \times\frac{1}{(2\pi)^M}\exp\left\{-\frac{2\kappa_0}{\alpha^2\gamma^{\alpha-1}k}\left\{1-\exp\left[-\frac{\alpha k}{\gamma}(t-t')\right]\right\}\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)\sum_{m=1}^{M}|b_m|^\alpha\right\} \times\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_{m=1}^{M}\exp\{i\mathcal{K}(\eta_m)+ib_m\eta_m\}d\eta_1\cdots d\eta_M. \quad (90)$$

By using the notations introduced in Sec. IV [Eqs. (49) and (50)], Eq. (90) can be rewritten in the form

$$\mathcal{G}[\mathcal{K}(x')] = \sum_{M=0}^{\infty} \frac{1}{M!} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathcal{M}_M(-b_1\xi, \ldots, -b_M\xi; t_0) db_1 \ldots db_M$$
$$\times \frac{1}{(2\pi)^M} \exp\left\{ -[\zeta^*(t-t_0)]^\alpha \sum_{m=1}^{M} |b_m|^\alpha \right\} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{m=1}^{M} \exp\{ i\mathcal{K}(\eta_m) + i b_m \eta_m \} d\eta_1 \cdots d\eta_M , \tag{91}$$

where $\zeta^*(t-t_0)$ is given by Eq. (49).

From Eq. (91) we can compute all stochastic properties of the concentration field. The moments $\langle C(x_1;t) \cdots C(x_{\mathcal{N}};t) \rangle$ are given by Eqs. (68) where $\zeta(t-t_0)$ is replaced by $\zeta^*(t-t_0)$. The cumulants of first and second order of the concentration are given by

$$\langle\langle C(x;t) \rangle\rangle = \int_{-\infty}^{+\infty} \frac{\langle\langle C_0(x) \rangle\rangle}{\zeta_\infty^*} \Psi_\alpha \left\{ \frac{x \exp[\nu(t-t_0)/\alpha] - y'}{\zeta_\infty^* \{\exp[\nu(t-t_0)] - 1\}^{1/\alpha}} \right\} dy' , \tag{92}$$

$$\langle\langle C(x_1;t) C(x_2;t) \rangle\rangle = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\langle\langle C_0(y_1') C_0(y_2') \rangle\rangle - \delta(y_1' - y_2') \langle\langle C_0(y_1') \rangle\rangle}{(\zeta_\infty^*)^2}$$
$$\times \prod_{u=1,2} \Psi_\alpha \left\{ \frac{x_u \exp[\nu(t-t_0)/\alpha] - y_u'}{\zeta_\infty^* \{\exp[\nu(t-t_0)] - 1\}^{1/\alpha}} \right\} dy_1' dy_2'$$
$$+ \delta(x_1 - x_2) \int_{-\infty}^{+\infty} \frac{\langle\langle C_0(y_1') \rangle\rangle}{\zeta_\infty^*} \Psi_\alpha \left\{ \frac{x_1 \exp[\nu(t-t_0)/\alpha] - y_1'}{\zeta_\infty^* \{\exp[\nu(t-t_0)] - 1\}^{1/\alpha}} \right\} dy_1' . \tag{93}$$

Similarly, the grand canonical joint probability densities, $\mathcal{C}_M(x_1, \ldots, x_M; t)$, the product densities $\mathcal{I}_m(x_1, \ldots, x_m; t)$, and the correlation functions $\mathscr{g}_m(x_1, \ldots, x_m; t)$ are given by

$$\mathcal{C}_M(x_1, \ldots, x_M; t) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \frac{\mathcal{C}_M^0(y_1', \ldots, y_M')}{(\zeta_\infty^*)^M} \prod_{w=1}^{M} \Psi_\alpha \left\{ \frac{x_w \exp[\nu(t-t_0)/\alpha] - y_w'}{\zeta_\infty^* \{\exp[\nu(t-t_0)] - 1\}^{1/\alpha}} \right\} dy_1' \cdots dy_N' , \tag{94}$$

$$\mathcal{I}_m(x_1, \ldots, x_m; t) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \frac{\mathcal{I}_m^0(y_1', \ldots, y_m')}{(\zeta_\infty^*)^m} \prod_{w=1}^{m} \Psi_\alpha \left\{ \frac{x_w \exp[\nu(t-t_0)/\alpha] - y_w'}{\zeta_\infty^* \{\exp[\nu(t-t_0)] - 1\}^{1/\alpha}} \right\} dy_1' \cdots dy_m' , \tag{95}$$

$$\mathscr{g}_m(x_1, \ldots, x_m; t) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \frac{g_m(y_1', \ldots, y_m'; t_0)}{(\zeta_\infty^*)^m} \prod_{w=1}^{m} \Psi_\alpha \left\{ \frac{x_w \exp[\nu(t-t_0)/\alpha] - y_w'}{\zeta_\infty^* \{\exp[\nu(t-t_0)] - 1\}^{1/\alpha}} \right\} dy_1' \cdots dy_m' . \tag{96}$$

The interpretation of the result derived in this section for systems with Lévy white noise is similar to the interpretation of the results derived in Sec. V for systems with static disorder. The many-body approach is consistent with the one-particle theory developed in Sec. IV. The relationships (78)–(88) for the probability density functional of concentration fluctuations for an initial Poissonian distribution remain valid, with the difference that for Lévy white noise the average concentration field $\langle\langle C(x;t) \rangle\rangle$ is given by Eq. (92).

VII. THERMODYNAMIC AND STOCHASTIC THEORY FOR LÉVY DIFFUSION IN A FORCE FIELD

In this section we use the results of the many-body approach derived in Secs. V and VI for developing a nonequilibrium thermodynamic approach for Lévy diffusion. We start out by investigating the large time behavior for an ensemble of Lévy particles in systems with static disorder. For large times the characteristic functional $\mathcal{G}[\mathcal{K}(x')]$ of the concentration field tends toward a stationary form. For large times Eq. (65) leads to

$$\lim_{t\to\infty} \mathcal{G}[\mathcal{K}(x')]$$
$$= \sum_{M=0}^{\infty} \frac{P(M; t=t_0)}{(2\pi\zeta_\infty)^M} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{m=1}^{M} \left[\Psi_\alpha \left(\frac{\eta_m}{\zeta_\infty} \right) \times \exp[i\mathcal{K}(\eta_m)] \right] d\eta_1 \cdots d\eta_M$$
$$= \sum_{M=0}^{\infty} P(M; t=t_0) \left[\int_{-\infty}^{+\infty} \Psi_\alpha \left(\frac{\eta}{\zeta_\infty} \right) \times \exp[i\mathcal{K}(\eta)] \frac{d\eta}{2\pi\zeta_\infty} \right]^M , \tag{97}$$

where we have used the obvious identity

$$P_0(M) = \frac{1}{M!} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathcal{M}_M(0, \ldots, 0; t=t_0) db_1 \cdots db_M , \tag{98}$$

and $P_0(M)$ is the probability that the system contains M particles for $t=t_0$.

We introduce the one-particle stochastic fractal potential $\mathcal{V}(x)$,

$$\Psi_\alpha\left(\frac{x}{\zeta_\infty}\right)\frac{dx}{\zeta_\infty}=\exp[-\mathcal{V}(x)]\frac{dx}{\zeta_\infty}$$

$$\text{that is}\quad \mathcal{V}(x)=-\ln\left[\Psi_\alpha\left(\frac{x}{\zeta_\infty}\right)\right], \tag{99}$$

and the generating function of the initial number of particles

$$\mathbb{G}(z)=\sum_{M=0}^{\infty} z^M P(M;t=t_0). \tag{100}$$

From Eqs. (97) and (100) we obtain

$$\lim_{t\to\infty}\mathcal{G}[\mathcal{K}(x')]=\mathbb{G}\left[\int_{-\infty}^{+\infty}\exp[i\mathcal{K}(\eta)-\mathcal{V}(\eta)]\frac{d\eta}{2\pi\zeta_\infty}\right]. \tag{101}$$

We denote by $\langle\langle M^m\rangle\rangle$ the cumulants of different orders of the initial number of particles. These cumulants are defined by the expansion

$$\ln\mathbb{G}[z=\exp(ib)]=\ln\left\{\sum_M \exp(ibM)P(M)\right\}$$
$$=\sum_{m=1}^{\infty}\frac{(ib)^m}{m!}\langle\langle M^m\rangle\rangle. \tag{102}$$

In this section we limit ourselves to systems for which the initial fluctuations of the number of particles are nonintermittent. For this type of system the relative fluctuations of different orders

$$\rho_m=\langle\langle M^m\rangle\rangle/\langle\langle M\rangle\rangle^m,\quad m=2,3,\ldots \tag{103}$$

tend towards zero in the thermodynamic limit

$$\rho_m=\langle\langle M^m\rangle\rangle/\langle\langle M\rangle\rangle^m\to 0,$$
$$m=2,3,\ldots\quad \text{as}\ \langle\langle M\rangle\rangle,\quad L\to\infty$$
$$\text{with}\ \langle\langle C_0\rangle\rangle=\langle\langle M\rangle\rangle/L\ \text{constant}, \tag{104}$$

where L is the linear dimension of the system.

In this limit Eq. (101) becomes

$$\lim_{t\to\infty}\mathcal{G}[\mathcal{K}(x')]=\exp\left\{\langle\langle M\rangle\rangle\left[\int_{-\infty}^{+\infty}\exp[i\mathcal{K}(\eta)-\mathcal{V}(\eta)]\frac{d\eta}{2\pi\zeta_\infty}-1\right]\right\}\quad \text{as}\ \langle\langle M\rangle\rangle\to\infty. \tag{105}$$

The corresponding probability density functional of concentration fluctuations has a Poissonian form similar to Eqs. (81) and (82)

$$\mathcal{P}[C(x)]\mathcal{D}[C(x)]$$
$$=\exp\left\{-\int_{-\infty}^{+\infty}C(x)\ln\left[\frac{\zeta_\infty C(x)}{\langle\langle M\rangle\rangle\Psi_\alpha(x/\zeta_\infty)}\right]dx\right\}\times\mathcal{D}[C(x)]$$
$$=\exp\left\{-\int_{-\infty}^{+\infty}C(x)\ln\left[\frac{\zeta_\infty C(x)}{\langle C(x)\rangle}\right]dx\right\}\mathcal{D}[C(x)]$$
$$=\exp\{-\Phi[C(x)]\}\mathcal{D}[C(x)], \tag{106}$$

where

$$\langle C(x)\rangle=\langle\langle M\rangle\rangle\frac{1}{\zeta_\infty}\Psi_\alpha\left(\frac{x}{\zeta_\infty}\right)=\frac{\langle\langle M\rangle\rangle}{\zeta_\infty}\exp[-\mathcal{V}(x)] \tag{107}$$

is the nonequilibrium stationary concentration profile corresponding to the one-particle stationary probability density (37) derived in Sec. IV,

$$\Phi[C(x)]=\int_{-\infty}^{+\infty}C(x)\ln\left[\frac{C(x)}{\langle C(x)\rangle}\right]dx$$
$$=\int_{-\infty}^{+\infty}C(x)\ln\left[\frac{\zeta_\infty C(x)}{\langle\langle M\rangle\rangle\Psi_\alpha(x/\zeta_\infty)}\right]dx$$
$$=\int_{-\infty}^{+\infty}C(x)\mathcal{V}(x)dx+\int_{-\infty}^{+\infty}C(x)\ln\left[\frac{\zeta_\infty C(x)}{\langle\langle M\rangle\rangle}\right]dx$$
$$=\mathcal{U}[C(x)]-\mathbb{S}[C(x)] \tag{108}$$

is a many-body stochastic potential, which is made up of the additive contribution of an energetic term

$$\mathcal{U}[C(x)]=\int_{-\infty}^{+\infty}C(x)\mathcal{V}(x)dx, \tag{109}$$

which depends on the one-particle nonequilibrium potential $\mathcal{V}(x)$, and on an entopic term

$$\mathbb{S}[C(x)]=-\int_{-\infty}^{+\infty}C(x)\ln\left[\frac{\zeta_\infty C(x)}{\langle\langle M\rangle\rangle}\right]dx. \tag{110}$$

We notice that the many-body stochastic potential $\Phi[C(x)]$ has a structure similar to Helmholtz or Gibbs free energies in equilibrium thermodynamics. Since concentration profile for large times corresponds to a nonequilibrium distribution, the one-particle stochastic potential $\mathcal{V}(x)$, which enters Eq. (109), is different from the harmonic mechanical potential defined by Eq. (3)

The many-body stochastic potential $\Phi[C(x)]$ may serve as the basis for the development of a nonequilibrium thermodynamic formalism for Lévy diffusion in external force fields. We consider an arbitrary variation $\delta C(x)$ of the concentration field. The conservation of the total number of particles requires that

$$\int_{-\infty}^{+\infty}\delta C(x)dx=0. \tag{111}$$

By taking the constraint (111) into account we can compute the first and the second variations of the stochastic potential, resulting in

$$\delta\Phi[C(x)] = \int_{-\infty}^{+\infty} \delta C(x) \ln\left[\frac{C(x)}{\langle C(x)\rangle}\right] dx,$$

$$\delta^2\Phi[C(x)] = \int_{-\infty}^{+\infty} \left\{\frac{[\delta C(x)]^2}{\langle C(x)\rangle}\right\} dx. \quad (112)$$

From Eqs. (112) it follows that the average concentration profile given by Eq. (107) corresponds to a minimum of the stochastic potential $\Phi[C(x)]$. We have

$$\delta\Phi[C(x) = \langle C(x)\rangle] = 0, \quad \delta^2\Phi[C(x)] > 0. \quad (113)$$

Equations (113) are similar to the conditions of existence and stability of thermodynamic equilibrium for normal (Fickian) diffusion in an external force field.

We can also introduce a field-chemical potential, which is made up of the additive contributions of the one-particle potential $\mathcal{V}(x)$ and of a "pure chemical contribution"

$$\begin{aligned}\tilde{\mu}[C(x)] &= \delta\Phi[C(x)]/\delta C(x) \\ &= \ln[C(x)/\langle C(x)\rangle] \\ &= \mathcal{V}(x) + \ln[\zeta_\infty C(x)/\langle\langle M\rangle\rangle]. \end{aligned} \quad (114)$$

All these results can be easily extended to the case of diffusion in systems with Lévy white noise. Equations (97)–(114) remain valid, with the difference that the length scale ζ_∞ must be replaced by the length scale ζ_∞^* defined by Eq. (50). An interesting feature of the diffusion with Lévy white noise is that in this case the average concentration field is the solution of a fractional Fokker-Planck equation

$$\frac{\partial}{\partial t} C(x;t) = \frac{\partial}{\partial x}\left[\frac{\nu}{\alpha} x C(x;t)\right] + D_{\text{fract}} \frac{\partial^\alpha}{\partial x^\alpha} C(x;t), \quad (115)$$

where the fractional diffusion coefficient

$$D_{\text{fract}} = \frac{2\kappa_0}{\alpha\gamma^\alpha} \Gamma(1-\alpha) \cos\left(\frac{\pi\alpha}{2}\right) \quad (116)$$

is related to the characteristic length scale ζ_∞^* by the relationship

$$\xi_\infty^* = \left(\frac{D_{\text{fract}}}{\nu}\right)^{1/\alpha} = \left[\frac{2\kappa_0}{\alpha^2 k \gamma^{\alpha-1}} \Gamma(1-\alpha) \cos\left(\frac{\pi\alpha}{2}\right)\right]^{1/\alpha}, \quad (117)$$

and the fractional derivative $\partial^\alpha/\partial x^\alpha$ is defined by an inverse Fourier transformation

$$\begin{aligned}\frac{\partial^\alpha}{\partial x^\alpha} C(x;t) &= -\mathcal{F}^{-1}[\overline{C}(q;t)|q|^\alpha] \\ &= \frac{-1}{2\pi} \int_{-\infty}^{+\infty} \overline{C}(q;t)|q|^\alpha \exp(-iqx) dq \\ &= \frac{\Gamma(1+\alpha)}{\pi} \sin\left(\frac{\pi\alpha}{2}\right) \int_{-\infty}^{+\infty} \frac{C(x';t)}{|x-x'|^{\alpha+1}} dx'. \end{aligned} \quad (118)$$

In Appendix D we show that the stochastic potential

$$\Phi[C(x;t)] = \int_{-\infty}^{+\infty} C(x;t) \ln[C(x;t)/C_\infty(x)] dx, \quad (119)$$

where

$$C_\infty(x) = \langle C(x)\rangle = \langle\langle M\rangle\rangle \frac{1}{\zeta_\infty^*} \Psi_\alpha\left(\frac{x}{\zeta_\infty^*}\right) = \frac{\langle\langle M\rangle\rangle}{\zeta_\infty^*} \exp[-\mathcal{V}(x)] \quad (120)$$

is a Lyapunov function for the stochastic evolution equation (115); that is, it satisfies the following conditions:

$$\Phi[C(x;t)] > 0 \quad \text{for } C(x;t) \neq C_\infty(x)$$

and (121)

$$\Phi[C(x;t)] = 0 \quad \text{for } C(x;t) = C_\infty(x),$$

and

$$\frac{d}{dt}\Phi[C(x;t)] < 0 \quad \text{for } C(x;t) \neq C_\infty(x)$$

and (122)

$$\frac{d}{dt}\Phi[C(x;t)] = 0 \quad \text{for } C(x;t) = C_\infty(x).$$

It follows that all transient solutions of the fractional diffusion equation (115) tend towards the stationary nonequilibrium form (120).

We conclude this section by investigating the possibilities of introducing a nonequilibrium temperature for the stationary states, which emerge in the limit of large times. We start by considering the equilibrium limit, which corresponds to $\alpha \to 2$. For $\alpha \to 2$ the one-particle probability density of the position of a moving particle has a Gaussian behavior, which corresponds to the equilibrium Maxwell-Boltzmann statistics. For both types of processes studied in this paper we have

$$P_{\text{eq}}(x) = \lim_{t\to\infty} P(x;t) = \frac{\sqrt{k}}{\sqrt{2\pi k_B T}} \exp\left[-\frac{kx^2}{2k_B T}\right], \quad (123)$$

and then

$$G_{\text{eq}}(q) = \int_{-\infty}^{+\infty} P_{\text{eq}}(x) \exp(iqx) dx = \exp\left\{-\frac{k_B T}{2k} q^2\right\}. \quad (124)$$

For an arbitrary value of the fractal exponent α the Fourier transform of the stationary one-particle probability density is given by a stretched exponential. We have

$$G_{\rm st}(q)=\int_{-\infty}^{+\infty}P_{\rm st}(x)\exp(iqx)dx=\exp[-(\zeta_\infty)^\alpha|q|^\alpha], \tag{125}$$

for systems with static disorder and

$$G_{\rm st}(q)=\int_{-\infty}^{+\infty}P_{\rm st}(x)\exp(iqx)dx=\exp[-(\zeta_\infty^*)^\alpha|q|^\alpha] \tag{126}$$

for dynamic disorder with Lévy white noise. By comparing Eq. (14) with Eqs. (125) and (126) we notice that the factors $(\zeta_\infty)^\alpha$ and $(\zeta_\infty^*)^\alpha$ play a role similar to the temperature in equilibrium thermodynamics. By considering suitable proportionality factors we can introduce a nonequilibrium temperature $T_{\rm noneq}$ which is proportional to $(\zeta_\infty)^\alpha$ or $(\zeta_\infty^*)^\alpha$ and which, for $\alpha\to2$, reduces to the equilibrium temperature. We have

$$T_{\rm noneq}=\frac{4k}{k_B}(\zeta_\infty)^\alpha=\frac{8k^{1-\alpha}\kappa}{\alpha k_B}\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right), \tag{127}$$

for systems with static disorder and

$$T_{\rm noneq}=\frac{4k}{k_B}(\zeta_\infty^*)^\alpha=\frac{8\kappa_0}{\alpha^2\gamma^{\alpha-1}k_B}\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right) \tag{128}$$

for dynamic disorder with white Lévy noise. In terms of the nonequilibrium temperature T we can introduce modified stochastic potentials which, for $\alpha\to2$, reduce to the thermodynamic functions of equilibrium thermodynamics. We have

$$\mathcal{P}[C(x)]\mathcal{D}[C(x)]=\exp\left\{-\frac{\Psi[C(x)]}{k_BT_{\rm noneq}}\right\}\mathcal{D}[C(x)], \tag{129}$$

where

$$\Psi[C(x)]=k_BT_{\rm noneq}\Phi[C(x)]$$
$$=\mathscr{h}[C(x)]-T_{\rm noneq}\mathcal{G}[C(x)], \tag{130}$$

is a stochastic potential which in the equilibrium limit, $\alpha\to2$, reduces to the Helmholtz free energy,

$$\mathscr{h}[C(x)]=k_BT_{\rm noneq}\mathcal{U}[C(x)] \tag{131}$$

is a function which in the equilibrium limit reduces to the total potential energy of the system and

$$\mathcal{G}[C(x)]=k_B\mathbb{S}[C(x)]/T_{\rm noneq} \tag{132}$$

is another function, which in the equilibrium limit reduces to the total entropy of the system. We emphasize that for a Lévy process with $\alpha\neq2$ the analogies between the modified stochastic potentials $\Psi[C(x)]$, $\mathscr{h}[C(x)]$, and $\mathcal{G}[C(x)]$, and their equilibrium limits corresponding to $\alpha\to2$ are rather limited. In the nonequilibrium regime it is not possible to build a fully developed thermodynamic formalism in terms of $\Psi[C(x)],\mathscr{h}[C(x)],\mathcal{G}[C(x)]$, and of the nonequilibrium temperature T.

VIII. CONCLUSIONS

In this paper we have suggested a method for studying Lévy diffusion in an external force field. We have used the Huber approach to multichannel parallel relaxation for describing the stochastic properties of the random force acting on a particle moving in a disordered system. Although the underlying physical mechanisms for multichannel parallel relaxation and Lévy diffusion are different, the corresponding evolution equations have a similar mathematical structure and the Huber approach can be easily extended to Lévy diffusion. We have investigated two extreme situations, a ''frozen'' random environment with static disorder and a system with strong dynamic disorder for which the fluctuations of the environment can be described in terms of Lévy white noise. For each case we have developed two different approaches, a one-particle description and a multiparticle description, respectively. Although the averaging techniques used and the detailed structure of the evolution equations are different for systems with static and dynamic disorder, the qualitative physical behavior is the same for both types of processes. In both cases the one-particle probability density of the position of the moving particle tends towards a nonequilibrium stationary form which is independent of the initial conditions and which reduces to an equilibrium Maxwell-Boltzmann distribution in the particular case where the Lévy fluctuations of the random force become Gaussian. In both cases the many-body approach makes it possible to compute the probability density functional of concentration fluctuations and to introduce a stochastic potential which is made up of the difference of an energetic and an entropic component. This stochastic potential extends the thermodynamic and stochastic theory of nonequilibrium processes by Ross, Hunt, and Hunt [16–20] to Lévy diffusion. For systems with fractal Lévy noise it is possible to introduce a fractional diffusion equation for which the stochastic potential is a Lyapunov function.

Other theories for Lévy diffusion presented in the literature are based on the Lévy generalization of the central limit theorem of probability theory. The transport process is viewed as a succession of a large number of individual jump processes and one assumes that the probability density of the length of a jump has infinite moments. By using the Lévy generalization of the central limit theorem it can be shown that a Lévy probability law for the propagator (Green function) of the process emerges for large successions of jumps and that this Lévy probability law is independent of the detailed form of the probability densities attached to the individual jumps. For these types of models Lévy diffusion occurs only asymptotically. In contrast, in the case of our model, the self-similar probability law (18) for the contribution g of an individual collision event to the total value of the random force leads precisely to a propagator (Green function) of the Lévy type. In our treatment the Lévy behavior of the Green functions is not asymptotic.

The many-body approaches developed in the present research deal with independent particles for which the correlations are due to the initial state of the system. Further developments of the theory should consider systems with interacting particles. A possible theoretical development may involve a perturbation theory for the characteristic functional for which the starting (reference) state is a system of independent particles like the ones studied in this paper.

In this paper we have introduced a powerful averaging technique that can be used for the study of a broad class of rate and transport processes in systems with static or dynamic disorder. An interesting type of system for which the method can be used is the study of reaction kinetics of a single molecule. Work on this problem is in progress and will be presented elsewhere.

ACKNOWLEDGMENTS

This research has been supported in part by the Alexander von Humboldt Foundation and by the Department of Energy, Basic Energy Sciences Engineering Program.

APPENDIX A

By inserting Eq. (65) into Eq. (66) and evaluating the functional derivatives we come to

$$\langle C(x_1;t)\cdots C(x_{\mathcal{N}};t)\rangle=\sum_{M=\mathcal{N}}^{\infty}\frac{1}{M!}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\mathcal{M}_M(-b_1\xi,\ldots,-b_M\xi;t)\frac{1}{(2\pi)^M}\exp\left\{-[\zeta(t-t_0)]^\alpha\sum_{m=1}^{M}|b_m|^\alpha\right\}db_1\cdots db_M$$
$$\times\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\sum_{\nu_1=1}^{M}\cdots\sum_{\nu_{\mathcal{N}}=1}^{M}\delta(\eta_{\nu_1}-x_1)\cdots\delta(\eta_{\nu_{\mathcal{N}}}-x_{\mathcal{N}})\prod_{m=1}^{M}\exp(ib_m\eta_m)d\eta_1\cdots d\eta_M. \tag{A1}$$

We use the identity

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\sum_{\nu_1=1}^{M}\cdots\sum_{\nu_{\mathcal{N}}=1}^{M}\delta(\eta_{\nu_1}-x_1)\cdots\delta(\eta_{\nu_{\mathcal{N}}}-x_{\mathcal{N}})\prod_{m=1}^{M}\exp(ib_m\eta_m)d\eta_1\cdots d\eta_M$$
$$=\sum_{\nu_1=1}^{M}\cdots\sum_{\nu_{\mathcal{N}}=1}^{M}\exp\left(i\sum_{u=1}^{\mathcal{N}}b_{\nu_u}x_u\right)\prod_{m\neq\nu_1,\ldots,\nu_{\mathcal{N}}}^{M}[2\pi\delta(b_m)], \tag{A2}$$

which is a straightforward consequence of the well-known relationship

$$2\pi\delta(q)=\int_{-\infty}^{+\infty}\exp(iqx)dx. \tag{A3}$$

From Eqs. (A1) and (A2) we get

$$\langle C(x_1;t)\cdots C(x_{\mathcal{N}};t)\rangle$$
$$=\sum_{M=\mathcal{N}}^{\infty}\frac{1}{M!}\sum_{\nu_1=1}^{M}\sum_{\nu_{\mathcal{N}_1}=1}^{M}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\mathcal{M}_M(-b_1\xi,\ldots,-b_M\xi;t)db_1\cdots db_M$$
$$\times\frac{1}{(2\pi)^{\mathcal{N}}}\exp\left\{-[\zeta(t-t_0)]^\alpha\sum_{m=1}^{M}|b_m|^\alpha\right\}\exp\left(i\sum_{u=1}^{\mathcal{N}}b_{\nu_u}x_u\right)\prod_{m\neq\nu_1,\ldots,\nu_{\mathcal{N}}}^{M}\delta(b_m)db_1\cdots db_M. \tag{A4}$$

By computing in Eq. (A4) $M-\mathcal{N}$ integrals in the Fourier variables b_m, $m=1,\ldots,M$, $m\neq\nu_1,\ldots,\nu_{\mathcal{N}}$ we come to Eq. (68).

For computing the grand canonical probability densities $\mathcal{C}(x_1,\ldots,x_M;t)$ we evaluate the multiple Fourier transforms of the moments of the concentration field

$$\mathcal{E}_{\mathcal{N}}(q_1,\ldots,q_{\mathcal{N}};t)=\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\exp\left(i\sum_{u=1}^{\mathcal{N}}q_ux_u\right)\langle C(x_1;t)\cdots C(x_{\mathcal{N}};t)\rangle dx_1\cdots dx_{\mathcal{N}}. \tag{A5}$$

We have

$$C(x_1;t)\cdots C(x_{\mathcal{N}};t)=\sum_{\nu_1=1}^{M}\cdots\sum_{\nu_{\mathcal{N}}=1}^{M}\prod_{u=1}^{\mathcal{N}}\delta(x'_{\nu_u}-x_u), \tag{A6}$$

and thus, if we assume that the ensemble averaging and the Fourier transformation commute, we get

$$
\begin{aligned}
&\mathcal{E}_{\mathcal{N}}(q_1,\ldots,q_{\mathcal{N}};t)\\
&=\left\langle \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} dx_1\cdots dx_{\mathcal{N}} \exp\left(i\sum_{u=1}^{\mathcal{N}} q_u x_u\right) \sum_{\nu_1=1}^{M}\cdots\sum_{\nu_{\mathcal{N}}=1}^{M}\prod_{u=1}^{\mathcal{N}} \delta(x'_{\nu_u}-x_u)\right\rangle\\
&=\left\langle \sum_{\nu_1=1}^{M}\cdots\sum_{\nu_{\mathcal{N}}=1}^{M} \exp\left[\sum_{u=1}^{\mathcal{N}} iq_u x'_{\nu_u}\right]\right\rangle\\
&=\sum_{M=\mathcal{N}}^{\infty}\frac{1}{M!}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\sum_{\nu_1=1}^{M}\cdots\sum_{\nu_{\mathcal{N}}=1}^{M} \exp\left[\sum_{u=1}^{\mathcal{N}} iq_u x'_{\nu_n}\right]\mathcal{C}_M(x'_1,\ldots,x'_M;t)dx'_1\ldots dx'_M\\
&=\sum_{M=\mathcal{N}}^{\infty}\frac{1}{M!}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} dx'_1\cdots dx'_M\sum_{\nu_1=1}^{M}\cdots\sum_{\nu_{\mathcal{N}}=1}^{M} \exp\left[\sum_{w=1}^{M}\sum_{u=1}^{\mathcal{N}} iq_u x'_{\nu_u}\delta_{w\nu_u}\right]\\
&\quad\times\frac{1}{(2\pi)^M}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\exp\left(-i\sum_{w=1}^{M} b_w x'_w\right)\mathcal{M}_M(b_1,\ldots,b_M;t)db_1\cdots db_M\\
&=\sum_{M=\mathcal{N}}^{\infty}\frac{1}{M!}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} dx'_1\cdots dx'_M\sum_{\nu_1=1}^{M}\cdots\sum_{\nu_{\mathcal{N}}=1}^{M}\\
&\quad\times\frac{1}{(2\pi)^M}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\exp\left[i\sum_{w=1}^{M}\left(\sum_{u=1}^{\mathcal{N}} iq_u\delta_{w\nu_u}-b_w\right)x'_w\right]\mathcal{M}_M(b_1,\ldots,b_M;t)db_1\cdots db_M.
\end{aligned}
\tag{A7}
$$

The multiple Fourier transform $\mathcal{E}_{\mathcal{N}}(q_1,\ldots,q_{\mathcal{N}};t)$ can be also computed from Eq. (68), resulting in

$$
\begin{aligned}
&\mathcal{E}_{\mathcal{N}}(q_1,\ldots,q_{\mathcal{N}};t)\\
&=\sum_{M=\mathcal{N}}^{\infty}\sum_{\nu_1=1}^{M}\cdots\sum_{\nu_{\mathcal{N}}=1}^{M}\frac{1}{M!}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\mathcal{M}_M^0(-b_1\xi,\ldots,b_M\xi)db_1\cdots db_M\\
&\quad\times\exp\left\{-[\zeta(t-t_0)]^{\alpha}\sum_{m=1}^{M}|b_m|^{\alpha}\right\}\prod_{m=1}^{M}\delta\left(\sum_{u=1}^{\mathcal{N}}\delta_{m\nu_u}q_u+b_m\right)\\
&=\sum_{M=\mathcal{N}}^{\infty}\sum_{\nu_1=1}^{M}\cdots\sum_{\nu_{\mathcal{N}}=1}^{M}\frac{1}{(2\pi)^M M!}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}dx'_1\cdots dx'_M\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}db_1\cdots db_M\mathcal{M}_M^0(-b_1\xi,\ldots,b_M\xi)\\
&\quad\times\exp\left\{-[\zeta(t-t_0)]^{\alpha}\sum_{m=1}^{M}|b_m|^{\alpha}+i\sum_{m=1}^{M}\left(\sum_{u=1}^{\mathcal{N}}\delta_{m\nu_u}q_u+b_m\right)x'_m\right\}.
\end{aligned}
\tag{A8}
$$

By comparing Eqs. (A7) and (A8) we notice that we must have

$$
\mathcal{M}_M(b_1,\ldots,b_M;t)=\mathcal{M}_M^0(b_1\xi,\ldots,b_M\xi)\exp\left\{-[\zeta(t-t_0)]^{\alpha}\sum_{m=1}^{M}|b_m|^{\alpha}\right\}.
\tag{A9}
$$

If we apply a multiple inverse transform to Eq. (A9) we come to Eq. (71).

APPENDIX B

The product densities defined by Eqs. (72) express the possibility of occurrence of m particles, the first at a position between x_1 and $x_1+dx_1,\ldots,$ and the mth at a position between x_m and x_m+dx_m, regardless of the numbers of particles that may exist at other positions. In Ref. [30] we have shown that the Carruthers definition of product densities [29] is equivalent to the classical definition of Stratonovich [26] and can be computed by using the following relations:

$$
\mathcal{I}_m(x_1,\ldots,x_m;t)=\sum_{M=m}^{\infty}\frac{1}{(M-m)!}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}dx_{m+1}\cdots dx_M\mathcal{C}_M(x_1,\ldots,x_M;t).
\tag{B1}
$$

By applying a multiple Fourier transformation to Eq. (B1) we obtain

$$\overline{\mathcal{I}}_m(b_1,\ldots,b_m;t)=\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\exp\left(i\sum_{u=1}^{m}x_u b_u\right)\mathcal{I}_m(x_1,\ldots,x_m;t)dx_1\cdots dx_m$$
$$=\sum_{M=m}^{\infty}\frac{1}{(M-m)!}\mathcal{M}_M(b_1,\ldots,b_m,b_{m+1}=0,\ldots,b_M=0;t), \tag{B2}$$

from which, by using Eq. (A9), we obtain

$$\overline{\mathcal{J}}_m(b_1,\ldots,b_m;t)=\exp\left\{-[\zeta(t-t_0)]^{\alpha}\sum_{u=1}^{m}|b_u|^{\alpha}\right\}$$
$$\times\sum_{M=m}^{\infty}\frac{1}{(M-m)!}\mathcal{M}_M(b_1\xi,\ldots,b_m\xi,b_{m+1}=0,\ldots,b_M=0;t_0)$$
$$=\exp\left\{-[\zeta(t-t_0)]^{\alpha}\sum_{u=1}^{m}|b_u|^{\alpha}\right\}\overline{\mathcal{I}}_m(b_1\xi,\ldots,b_m\xi;t_0). \tag{B3}$$

By applying a multiple inverse transformation to Eq. (B3) we get Eq. (73).

For computing the correlation functions $\mathscr{g}_m(x_1,x_2,\ldots,x_m;t)$ we introduce the generating functional $\mathcal{R}[\Theta(x);t]$ of the product densities

$$\mathcal{R}[\Theta(x);t]=1+\sum_{m=1}^{\infty}\frac{1}{m!}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\mathcal{I}_m(x_1,\ldots,x_M;t)\Theta(x_1)\cdots\Theta(x_m)dx_1\cdots dx_m, \tag{B4}$$

where $\Theta(x)$ is a suitable test function. We insert Eqs. (73) into Eq. (B4), resulting in

$$\mathcal{R}[\Theta(x);t]=1+\sum_{m=1}^{\infty}\frac{1}{m!}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\mathcal{I}_m^0(y_1',\ldots,y_m')$$
$$\times\prod_{w=1}^{m}\left\{\int_{-\infty}^{+\infty}\frac{\Theta(x_w)}{\zeta_\infty}\Psi_\alpha\left\{\frac{x_w\exp[(t-t_0)k/\gamma]-y_w'}{\zeta_\infty\{\exp[(t-t_0)k/\gamma]-1\}}\right\}dx_w\right\}dy_1'\cdots dy_m'$$
$$=\mathcal{R}\left[\Theta(y)=\int_{-\infty}^{+\infty}\frac{\Theta(x)}{\zeta_\infty}\Psi_\alpha\left\{\frac{x\exp[(t-t_0)k/\gamma]-y}{\zeta_\infty\{\exp[(t-t_0)k/\gamma]-1\}}\right\}dx;t_0\right]. \tag{B5}$$

The correlation functions $\mathscr{g}_m(x_1,x_2,\ldots,x_m;t)$ are defined by the cumulant expansion

$$\ln\mathcal{R}[\Theta(x);t]=\sum_{m=1}^{\infty}\frac{1}{m!}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\mathscr{g}_m(x_1,x_2,\ldots,x_m;t)\Theta(x_1)\cdots\Theta(x_m)dx_1\cdots dx_m. \tag{B6}$$

We take the logarithm of each term of Eq. (B5) and expand both terms by using the cumulant expansion (B6). We get

$$\sum_{m=1}^{\infty}\frac{1}{m!}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}dx_1\cdots dx_m\Theta(x_1)\cdots\Theta(x_m)$$
$$\times\left[\mathscr{g}_m(x_1,x_2,\ldots,x_m;t)-\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}dy_1'\,dy_N'\frac{\mathscr{g}_m(y_1',\ldots,y_m';t_0)}{(\zeta_\infty)^m}\prod_{w=1}^{M}\Psi_a\left\{\frac{x_w\exp[(t-t_0)k/\gamma]-y_w'}{\zeta_\infty\{\exp[(t-t_0)k/\gamma]-1\}}\right\}\right]. \tag{B7}$$

Since the test functions $\Theta(x_1),\ldots,\Theta(x_m)$ are arbitrary, it follows that the coefficients of the products $\Theta(x_1)\cdots\Theta(x_m)$ must equal zero, resulting in Eqs. (75).

APPENDIX C

By inserting Eq. (78) into Eq. (79) we obtain

$$\mathcal{G}[\mathcal{K}(x')]=\sum_{M=0}^{\infty}\frac{1}{M!}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\exp(-\langle M\rangle)\langle\overline{C}_0(-b_1\xi)\rangle\cdots\langle\overline{C}_0(-b_M\xi)\rangle db_1\cdots db_M$$
$$\times\frac{1}{(2\pi)^M}\exp\left\{-[\zeta(t-t_0)]^\alpha\sum_{m=1}^{M}|b_m|^\alpha\right\}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_{m=1}^{M}\exp\{i\mathcal{K}(\eta_m)+ib_m\eta_m\}d\eta_1\cdots d\eta_M$$
$$=\exp\left(-\langle M\rangle+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\langle\overline{C}_0(-b\xi)\rangle\exp\{i\mathcal{K}(\eta)+ib\eta-[\zeta(t-t_0)]^\alpha|b|^\alpha\}d\eta\,db\right),\tag{C1}$$

where

$$\langle\overline{C}_0(q)\rangle=\int_{-\infty}^{+\infty}\exp(iqx)\langle C_0(x)\rangle dx\tag{C2}$$

is the Fourier transform of the average concentration field at initial time. In Eq. (C1) the double integral in the exponent can be expressed in terms of the average concentration field at time t, $\langle C(x;t)\rangle=\langle\langle C(x;t)\rangle\rangle$. We have

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\langle\overline{C}_0(-b\xi)\rangle\exp\{i\mathcal{K}(\eta)+ib\eta-[\zeta(t-t_0)]^\alpha|b|^\alpha\}d\eta\,db$$
$$=\int_{-\infty}^{+\infty}\exp\{i\mathcal{K}(\eta)\}\mathcal{F}_\eta^{-1}\{\langle\overline{C}_0(-b\xi)\rangle\exp\{-[\zeta(t-t_0)]^\alpha|b|^\alpha\}\}d\eta,\tag{C3}$$

where $\mathcal{F}_\eta^{-1}$ represents the inverse Fourier transformation from the Fourier space variable $-b$ to the real space variable η. By using the convolution theorem and comparing the result with Eq. (69) we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\langle\overline{C}_0(-b\xi)\rangle\exp\{i\mathcal{K}(\eta)+ib\eta-[\zeta(t-t_0)]^\alpha|b|^\alpha\}d\eta\,db$$
$$=\int_{-\infty}^{+\infty}\exp\{i\mathcal{K}(\eta)\}\langle C(\eta;t)\rangle dx.\tag{C4}$$

We insert Eq. (C4) into Eq. (C1) and use Eq. (77), resulting in Eq. (80).

APPENDIX D

The derivation of Eq. (120) is based on the algebraic inequality

$$x\ln x-x+1>0\quad\text{for}\quad x\neq1\quad\text{and}\quad x\ln x-x+1=0\quad\text{fpr}\quad x=1.\tag{D1}$$

The fractional diffusion equation (115) preserves the conservation of the total number of particles. By using Eq. (118), we can rewrite the fractional diffusion equation (115) in the form

$$\frac{\partial}{\partial t}C(x;t)=\frac{\partial}{\partial x}\left[\frac{\nu}{\alpha}xC(x;t)\right]-\frac{D_{\rm fract}}{2\pi}\int_{-\infty}^{+\infty}dq\int_{-\infty}^{+\infty}dx'|q|^\alpha C(x';t)\exp[iq(x'-x)].\tag{D2}$$

We integrate Eq. (D2) term by term, with respect to the position of the particle, resulting in

$$\int_{-\infty}^{+\infty}dx\frac{\partial}{\partial t}C(x;t)=\int_{-\infty}^{+\infty}dx\frac{\partial}{\partial x}\left[\frac{\nu}{\alpha}xC(x;t)\right]-\frac{D_{\rm fract}}{2\pi}\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dq\int_{-\infty}^{+\infty}dx'|q|^\alpha C(x';t)$$
$$\times\exp[iq(x'-x)],\tag{D3}$$

from which we come to

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty}C(x;t)dx=-D_{\rm fract}\int_{-\infty}^{+\infty}dq\int_{-\infty}^{+\infty}dx'|q|^\alpha C(x';t)\delta(q)=0,\tag{D4}$$

and therefore

$$\int_{-\infty}^{+\infty} C(x;t)dx = \int_{-\infty}^{+\infty} C_\infty(x)dx. \tag{D5}$$

By using Eq. (D5) we can express the stochastic potential $\Phi[C(x;t)]$ in the form

$$\Phi[C(x;t)] = \int_{-\infty}^{+\infty} C(x;t)\left\{\ln\left[\frac{C(x;t)}{C_\infty(x)}\right] - C(x;t) + C_\infty(x)\right\}dx. \tag{D6}$$

From Eqs. (D1) and (D6) we come to Eqs. (121).

The derivation of Eqs. (122) is more complicated. We start by applying the fractional diffusion equation (D2) to the stationary concentration profile $C_\infty(x) = \langle C(x)\rangle$ given by Eq. (120)

$$0 = \frac{\partial}{\partial x}\left[\frac{\nu}{\alpha} x C_\infty(x)\right] - \frac{D_{\mathrm{fract}}}{2\pi}\int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dx' |q|^\alpha C_\infty(x') \exp[iq(x'-x)]. \tag{D7}$$

By using Eqs. (D2) and (D7) we can express the time derivative of the stochastic potential $\Phi[C(x;t)]$ in the following form:

$$\begin{aligned}\frac{d}{dt}\Phi[C(x;t)] &= \int_{-\infty}^{+\infty}\frac{\partial}{\partial t}[C(x;t)]\ln\left[\frac{C(x;t)}{C_\infty(x)}\right]dx + \int_{-\infty}^{+\infty}\frac{\partial}{\partial t}[C(x;t)]dx \\ &= \int_{-\infty}^{+\infty}\frac{\partial}{\partial t}[C(x;t)]\ln\left[\frac{C(x;t)}{C_\infty(x)}\right]dx = \int_{-\infty}^{+\infty}\left\{\frac{\partial}{\partial x}\left[\frac{\nu}{\alpha}xC(x;t)\right]\right\}\ln\left[\frac{C(x;t)}{C_\infty(x)}\right]dx \\ &\quad - \frac{D_{\mathrm{fract}}}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dq\int_{-\infty}^{+\infty}dx'|q|^\alpha C(x;t) \\ &\quad \times \exp[iq(x'-x)]\ln\left[\frac{C(x;t)}{C_\infty(x)}\right]dx.\end{aligned} \tag{D8}$$

The first integral in Eq. (D8) can be computed step by step. We have

$$\begin{aligned}\int_{-\infty}^{+\infty}\partial_x\left[\frac{\nu}{\alpha}xC(x;t)\right]\ln\left[\frac{C(x;t)}{C_\infty(x)}\right]dx &= \frac{\nu}{\alpha}xC(x;t)\ln\left[\frac{C(x;t)}{C_\infty(x)}\right]\Bigg|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\frac{\nu}{\alpha}xC(x;t)\frac{\partial}{\partial x}\left(\ln\left[\frac{C(x;t)}{C_\infty(x)}\right]\right)dx \\ &= -\int_{-\infty}^{+\infty}\frac{\nu}{\alpha}xC(x;t)\left[\frac{\partial_x C(x;t)}{C(x;t)} - \frac{\partial_x C_\infty(x)}{C_\infty(x)}\right]dx \\ &= -\int_{-\infty}^{+\infty}\frac{\nu}{\alpha}x\frac{\partial}{\partial x}C(x;t)dx + \int_{-\infty}^{+\infty}\frac{\nu}{\alpha}xC(x;t)\frac{\partial_x C_\infty(x)}{C_\infty(x)}dx \\ &= -\frac{\nu}{\alpha}xC(x;t)\Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty}\frac{\nu}{\alpha}C(x;t)dx + \int_{-\infty}^{+\infty}\frac{\nu}{\alpha}x\frac{C(x;t)}{C_\infty(x)}\frac{\partial}{\partial x}C_\infty(x)dx.\end{aligned} \tag{D9}$$

We assume vanishing boundary conditions

$$\lim_{x\to\pm\infty} xC(x;t) = 0. \tag{D10}$$

Note that for a concentration profile of the Lévy type we have $C(x,t) \sim |x|^{-(\alpha+1)}$ as $|x| \to \infty$ with $\alpha > 0$ and thus this condition is automatically fulfilled. We get

$$\int_{-\infty}^{+\infty}\partial_x\left[\frac{\nu}{\alpha}xC(x;t)\right]\ln\left[\frac{C(x;t)}{C_\infty(x)}\right]dx$$

$$=\int_{-\infty}^{+\infty}\frac{\nu}{\alpha}C(x;t)dx+\int_{-\infty}^{+\infty}\frac{C(x;t)}{C_\infty(x)}\frac{\nu}{\alpha}x\frac{\partial}{\partial x}C_\infty(x)dx$$

$$=\int_{-\infty}^{+\infty}\frac{\nu}{\alpha}C(x;t)dx+\int_{-\infty}^{+\infty}\frac{C(x;t)}{C_\infty(x)}\left\{-\frac{\nu}{\alpha}C_\infty(x)+\frac{D_{\text{fract}}}{2\pi}\int_{-\infty}^{+\infty}dq\int_{-\infty}^{+\infty}dx'|q|^\alpha C_\infty(x')\exp[iq(x'-x)]\right\}dx$$

$$=\frac{D_{\text{fract}}}{2\pi}\int_{-\infty}^{+\infty}dx\frac{C(x;t)}{C_\infty(x)}\int_{-\infty}^{+\infty}dq\int_{-\infty}^{+\infty}dx'|q|^\alpha C_\infty(x')\exp[iq(x'-x)]. \tag{D11}$$

By combining Eqs. (D8) and (D11) we obtain

$$\frac{d}{dt}\Phi[C(x;t)]=\frac{D_{\text{fract}}}{2\pi}\int_{-\infty}^{+\infty}dx\frac{C(x;t)}{C_\infty(x)}\int_{-\infty}^{+\infty}dq\int_{-\infty}^{+\infty}dx'|q|^\alpha C_\infty(x')\exp[iq(x'-x)]$$
$$-\frac{D_{\text{fract}}}{2\pi}\int_{-\infty}^{+\infty}dx\ln\left[\frac{C(x;t)}{C_\infty(x)}\right]\int_{-\infty}^{+\infty}dq\int_{-\infty}^{+\infty}dx'|q|^\alpha C(x';t)\exp[iq(x'-x)], \tag{D12}$$

from which we come to

$$\frac{d}{dt}\Phi[C(x;t)]$$
$$=\frac{D_{\text{fract}}}{2\pi}\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dq\int_{-\infty}^{+\infty}dx'|q|^\alpha\left\{C_\infty(x')\frac{C(x;t)}{C_\infty(x)}-C(x';t)\ln\left[\frac{C(x;t)}{C_\infty(x)}\right]\right\}\exp[iq(x'-x)]$$
$$=\frac{D_{\text{fract}}}{2\pi}\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dq\int_{-\infty}^{+\infty}dx'|q|^\alpha C(x';t)\left\{\frac{C_\infty(x')C(x;t)}{C(x';t)C_\infty(x)}-\ln\left[\frac{C(x;t)}{C_\infty(x)}\right]\right\}\exp[iq(x'-x)]. \tag{D13}$$

We use the identity

$$\frac{C(x;t)C_\infty(x')}{C_\infty(x)C(x';t)}-\ln\left[\frac{C(x;t)}{C_\infty(x)}\right]=\frac{C(x;t)C_\infty(x')}{C_\infty(x)C(x';t)}-\ln\left[\frac{C(x;t)C_\infty(x')}{C_\infty(x)C(x';t)}\right]-1+\ln\left[\frac{C_\infty(x')}{C(x';t)}\right]+1. \tag{D14}$$

By combining Eqs. (D13) and (D14) we can express the time derivative of the stochastic potential as the sum of two different terms. We have

$$\frac{d}{dt}H[C(x;t)]=T_1+T_2, \tag{D15}$$

where

$$T_1=\frac{D_{\text{fract}}}{2\pi}\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dq\int_{-\infty}^{+\infty}dx'|q|^\alpha C(x';t)\left\{\frac{C_\infty(x')C(x;t)}{C(x';t)C_\infty(x)}-\ln\left[\frac{C_\infty(x')C(x;t)}{C(x';t)C_\infty(x)}\right]-1\right\}\exp[iq(x'-x)]$$
$$=D_{\text{fract}}\frac{\Gamma(1+\alpha)}{\pi}\sin\left(\frac{\pi\alpha}{2}\right)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{C(x';t)}{|x-x'|^{\alpha+1}}\left\{\ln\left[\frac{C(x;t)C_\infty(x')}{C_\infty(x)C(x';t)}\right]-\frac{C(x;t)C_\infty(x')}{C_\infty(x)C(x';t)}+1\right\}dx\,dx' \tag{D16}$$

and

$$T_2=\frac{D_{\text{fract}}}{2\pi}\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dq\int_{-\infty}^{+\infty}dx'|q|^\alpha C(x';t)\left[\ln\left(\frac{C_\infty(x')}{C(x';t)}\right)+1\right]\exp[iq(x'-x)]. \tag{D17}$$

In order to compute the term T_2 we use the identity

$$\delta(q)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp(iqy)dy. \tag{D18}$$

We come to

$$T_4 = D_{\text{fract}} \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dx' |q|^{\alpha} C(x';t) \left\{ \ln\left[\frac{C_\infty(x')}{C(x';t)} \right] + 1 \right\} \delta(q) = 0, \tag{D19}$$

and therefore the time derivative of the stochastic potential is given by

$$\frac{d}{dt} \Phi[C(x;t)] = D_{\text{fract}} \frac{\Gamma(1+\alpha)}{\pi} \sin\left(\frac{\pi\alpha}{2}\right) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{C(x';t)}{|x-x'|^{\alpha+1}} \left\{ \ln\left[\frac{C(x;t)C_\infty(x')}{C_\infty(x)C(x';t)} \right] - \frac{C(x;t)C_\infty(x')}{C_\infty(x)C(x';t)} + 1 \right\} dx\, dx'. \tag{D20}$$

From Eqs. (116) and from the well-known algebraic identity

$$\Gamma(1+\alpha)\Gamma(1-\alpha) = \frac{\alpha\pi}{\sin(\pi\alpha)} \tag{D21}$$

it turns out that

$$D_{\text{fract}} \frac{\Gamma(1+\alpha)}{\pi} \sin\left(\frac{\pi\alpha}{2}\right) = \frac{\kappa_0}{\gamma^{\alpha}} > 0. \tag{D22}$$

From Eqs. (D20) and (D22) and from the algebraic inequality

$$\ln x - x + 1 < 0 \quad \text{for} \quad x \neq 1 \quad \text{and} \quad \ln x - x + 1 = 0 \quad \text{for} \quad x = 1, \tag{D23}$$

we come to Eq. (122).

---

[1] H. Scher and E. W. Montroll, Phys. Rev. B **12**, 2455 (1974); E. W. Montroll and B. J. West, in *Fluctuation Phenomena, Studies on Statistical Mechanics*, edited by E. W. Montroll and J. L. Lebowitz (North-Holland, Amsterdam, 1979), Vol. 7, pp. 63–175.

[2] S. Alexander, J. Bernasconi, R. W. Schneider, and R. Orbach, Rev. Mod. Phys. **53**, 175 (1981).

[3] S. Havlin and D. Ben-Avraham, Adv. Phys. **36**, 695 (1987).

[4] J. P. Bouchaud and A. Georges, Phys. Rep. **196**, 127 (1990).

[5] R. A. Daimon and K. J. Packer, Proc. R. Soc. London, Ser. A **453**, 205 (1997).

[6] A. Blumen, J. Klafter, and G. Zumofen, in *Optical Spectroscopy of Glasses*, edited by I. Zschokke (Reidel, Amsterdam, 1986), pp. 199–265; A. Plonka, *Time-Dependent Reactivity of Species in Condensed Media*, Lecture Notes on Chemistry Vol. 40 (Springer, Berlin, 1986); Prog. React. Kinet. **16**, 157 (1991); Annu. Rep. Prog. Chem., Sect. C: Phys. Chem. **89**, 37 (1992).

[7] M. F. Shlesinger, Annu. Rev. Phys. Chem. **39**, 269 (1988).

[8] B. J. West and W. Deering, Phys. Rep. **246**, 1 (1994); B. J. West, Int. J. Mod. Phys. B **10**, 67 (1996); Physica A **230**, 359 (1996).

[9] M. O. Vlad, C. Lacoursière, and M. C. Mackey, Phys. Lett. A **208**, 99 (1995).

[10] J. Klafter, A. Blumen, and M. F. Shlesinger, Phys. Rev. A **35**, 3081 (1987).

[11] B. J. West and V. Seshadri, Physica A **113**, 203 (1982); F. Pesekis, Phys. Rev. A **36**, 892 (1987).

[12] H. C. Fogedby, Phys. Rev. Lett. **73**, 2157 (1994); Phys. Rev. E **50**, 1654 (1994); **58**, 1690 (1998).

[13] S. Jespersen, R. Metzler, and H. C. Fogedby, Phys. Rev. E **59**, 2736 (1999).

[14] R. Metzler, E. Bakrai, and J. Klafter, Europhys. Lett. **46**, 431 (1999); Phys. Rev. Lett. **82**, 3563 (1999); E. Barkai, R. Metzler, and J. Klafter, Phys. Rev. E **61**, 132 (2000).

[15] M. O. Vlad, J. Ross, and D. L. Huber, J. Phys. Chem. **103**, 1563 (1999).

[16] J. Ross, K. C. L. Hunt, and P. M. Hunt, J. Chem. Phys. **88**, 2179 (1998); **96**, 618 (1992).

[17] J. Ross, R. H. Harding, A. N. Wolff, and X. Chu, J. Chem. Phys. **95**, 5206 (1991); J. Ross, X. Chu, A. Hjelmfeldt, and M. Velarde, J. Phys. Chem. **97**, 4708 (1993).

[18] Bo Peng, K. L. C. Hunt, P. M. Hunt, A. Suàrez, and J. Ross, J. Chem. Phys. **102**, 4548 (1995).

[19] A. Suàrez, J. Ross, Bo Peng, K. L. C. Hunt, and P. M. Hunt, J. Chem. Phys. **102**, 4563 (1995); A. Suàrez and J. Ross, J. Phys. Chem. **99**, 14 854 (1995); W. Vance and J. Ross, J. Chem. Phys. **105**, 479 (1996).

[20] M. O. Vlad and J. Ross, J. Chem. Phys. **100**, 7268 (1994); **100**, 7279 (1994); **100**, 7295 (1994).

[21] D. L. Huber, Phys. Rev. B **31**, 6070 (1985).

[22] M. O. Vlad and M. C. Mackey, J. Math. Phys. **36**, 1834 (1995).

[23] M. O. Vlad, B. Schönfisch, and M. C. Mackey, Phys. Rev. E **53**, 4703 (1996).

[24] D. L. Huber, Phys. Rev. E **53**, 6544 (1996); M. O. Vlad, J. Ross, and M. C. Mackey, J. Math. Phys. **37**, 803 (1996); M. O. Vlad, R. Metzler, T. Nonnenmacher, and M. C. Mackey, *ibid.* **37**, 2279 (1996).

[25] A. Ramakrishnan, in *Encyclopedia of Physics*, edited by S. Függe (Springer, Berlin, 1959), Vol. III/2, pp. 525–651

[26] R. L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1963), Vol. 1.

[27] D. L Snyder, *Random Point Processes* (Wiley, New York, 1975); D. R. Cox and V. Isham, *Point Processes* (Chapman and Hall, London, 1980).

[28] E. W. Montroll and J. T. Bendler, J. Stat. Phys. **34**, 129 (1984).

[29] P. Carruthers, Phys. Rev. A **43**, 2632 (1991).

[30] M. O. Vlad and M. C. Mackey, Phys. Rev. E **51**, 3104 (1995).